Affine 空間 : リーマン空間における拡張されたJacobian接続径数について

メタデータ	言語: English 出版者: 公開日: 2012-11-07 キーワード (Ja): キーワード (En): 作成者: 叶, 長太郎, 中川, 正 メールアドレス: 所属:
URL	https://doi.org/10.32150/00001032

Vol.14 No.2 Journal of Hokkaido Gakugei University (Section IIA) Dec., 1963

On the Extended Jacobian Connection Parameter in a Space with Affine Connection and in a Riemannian Space

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Introduction. The Jacobian extensor introduced by H. V. Craig appears as an extension of the tensor concept of the weight W and holds tensor member having weight as a part of its components. On the other hand Y. Katurada established an excovariant derivation of an extensor in a space with an affine connection without torsion as well as in a Riemannian space, using the extended connection parameter $\Gamma^{\alpha a}_{\beta b\ \gamma c}$ formed by the affine connection parameter Γ^{a}_{bc} and Christoffel symbol by means of the metric extensor $g_{\alpha a\,\beta b}$ induced by the metric tensor g_{ab} respectiively.

The principal purpose of the present paper is to establis an excovariant derivation of a Jacobian extensor in a space with an object Q_a or an affine connection, and in a Riemannian space.

In the present paper we use certain of the ideas, notations and results given in the Y. Katurada's and Craig's paper without explanation.

§ *1.* *The excovariant derivation of the Jacobian extensor.*

We shall proceed to find what is so called the extended Jacobian connection parameter and to build up such the Jacobian excovariant differentiation that contains the covariant differential of a tensor having weight as a part of it, in the space with an affine connection.

Let us given a quantity $Q_c(t)$ at each point on a parameterized arc of class M, where t is a fixed essential parameter, changed by the following transformation equation:

(1.1) $$Q_c(x(t)) = Q_r(\bar{x}(t))X^r_c + X^r_{ca}X^a_r w,$$

where

$$wX^r_{ca}X^c_r = w\frac{\partial^2 \bar{x}^r}{\partial x^c \partial x^a}\frac{\partial x^c}{\partial \bar{x}^r} = \partial\, log(\bar{x}\, x)^w / \partial x^a$$

We can then state the following thorem:

Theorem 1.1 *The quantities $Q^{\delta}_{\alpha\ \beta c}$ are changed by*

(1.2) $$Q^{\delta}_{\alpha\ \beta c} = \sum_{\theta=\beta}^{\delta-\alpha} Q^{\delta}_{\alpha\ \theta r}X^{\theta r}_{\beta c} + \binom{\delta}{\alpha\beta}\left(\frac{\partial\, log\,\bar{x}}{\partial x^c}\right)^{(\delta-\alpha-\beta)}$$

under a transformation, of the expoint ([1] p. 765), *where*

$$Q^{\delta}_{\alpha\ \beta c} = \binom{\delta}{\alpha\beta} Q^{(\delta-\alpha-\beta)} \qquad \textit{for } \alpha+\beta \leqq \delta,$$
$$= 0 \qquad \textit{for } \alpha+\beta > \delta.$$

Proof. Differentiating the transformation equation (1.1) $(\delta-a-\beta)$ times by t and multiplying the generalized binomial coefficient $\binom{\delta}{\alpha\beta}$, the left-hand member is equal to $Q^{\delta}_{\alpha\ \beta c}$, and the first terms of the right hand member becomes as follows:

$$\binom{\delta}{\alpha\beta}(Q_r X^r_c)^{(\delta-\alpha-\beta)} = \sum_{\varphi} \binom{\delta}{\alpha\beta} \binom{\delta-\alpha-\beta}{\varphi} (Q_r)^{(\delta-\alpha-\beta-\varphi)} (X^r_c)^{(\varphi)}$$
$$= \sum_{\theta} \binom{\delta}{\alpha\beta} \binom{\delta-\alpha-\beta}{\theta-\beta} \binom{\theta}{\beta}^{-1} Q_r^{(\delta-\alpha-\theta)} X^{r\theta}_{\beta c}$$
$$= \sum_{\theta} \binom{\delta}{\alpha\theta} Q_r^{(\delta-\alpha-\theta)} X^{r\theta}_{\beta c},$$

accordingly (1.2) is obtained.

The differential of a Jacobian extensor V^{ρ} is changed by the following equation:

(1.3) $$dV^{\rho} = dX^{\rho}_{\alpha} V^{\alpha} + X^{\rho}_{\alpha} V^{\alpha}$$

where

(1.4) $$dX^{\rho}_{\alpha} = \sum_{\beta} X^{\rho}_{\alpha;\ \beta c} dx^{\beta c}$$
$$= \sum_{\beta} \binom{\rho}{\alpha} \overline{X}^{(\rho-\alpha)}{}_{;\ \beta c}\, dx^{\beta c}$$
$$= \sum_{\beta} \binom{\rho}{\alpha} \binom{\rho-\alpha}{\beta} \overline{X}^{(\rho-\alpha-\beta)}{}_{;\ c} dx^{\beta c}$$
$$= \sum_{\beta} \binom{\rho}{\alpha} \binom{\rho-\alpha}{\beta} \left(\overline{X} \frac{\partial \log \overline{X}}{\partial x^c}\right)^{(\rho-\alpha-\beta)} dx^{\beta c}$$
$$= \sum_{\beta} \binom{\rho}{\alpha} \binom{\rho-\alpha}{\beta} \sum_{\varphi} \binom{\rho-\alpha-\beta}{\varphi} \overline{X}^{(\varphi)} \left(\frac{\partial \log \overline{X}}{\partial x^c}\right)^{(\rho-\alpha-\beta-\varphi)} dx^{\beta c}$$
$$= \sum_{\beta} \binom{\rho}{\alpha} \binom{\rho-\alpha}{\beta} \sum_{\delta} \binom{\rho-\alpha-\beta}{\rho-\delta} \overline{X}^{(\rho-\alpha)} \left(\frac{\partial \log \overline{X}}{\partial x^c}\right)^{(\delta-\alpha-\beta)} dx^{\beta c}$$
$$(\textit{putting } \varphi = p-\delta)$$
$$= \sum_{\beta} \sum_{\delta} \binom{\rho}{\alpha} \binom{\rho-\alpha}{\beta} \binom{\rho-\alpha-\beta}{\rho-\delta} \binom{\rho}{\delta}^{-1} X^{\rho}_{\delta} \left(\frac{\partial \log \overline{X}}{\partial x^c}\right)^{(\delta-\alpha-\beta)} dx^{\beta c}$$
$$= \sum_{\beta} \sum_{\delta} X^{\rho}_{\delta} \binom{\delta}{\alpha\beta} \left(\frac{\partial \log \overline{X}}{\partial x^c}\right)^{(\delta-\alpha-\beta)} dx^{\beta c}.$$

From (1.2), (1.3) and (1.4) we have

$$\bar{\delta} V^{\rho} = \sum_{\delta} X^{\rho}_{\delta}\ \ \delta V^{\delta},$$

where

$$\bar{\delta} V^{\delta} = dV^{\delta} + \sum\sum Q^{\delta}_{\alpha\ \beta c} V^{\alpha} dx^{\beta c}.$$

Since the quantities $w\Gamma^{a}_{a c}$ and $w\Gamma^{a}_{c a}$ formed from an affine connection parameter $\Gamma^{a}_{b c}$ is examples of the quantities Q_c, then we shall call the extended Jacobian connection parameters the objects $Q^{\delta}_{\beta c\ \alpha}$ and $Q^{\delta}_{\alpha\ \beta c}$ which is derived from $w\Gamma^{a}_{a\ c}$ and $w\Gamma^{a}_{c\ a}$, and write $\Gamma^{\delta}_{\alpha\ \beta c}$ and $\Gamma^{\delta}_{\beta c}$ respectively. Then we have easily the theorems.

Theorem 1.2 *$\Gamma^{\delta}_{\alpha\ \beta c}$ and $\Gamma^{\delta}_{\beta c\ \alpha}$ is symmetric with respect to the indices α and β.*

Theorem 1.3 *When the index δ of $\Gamma^{\delta}_{\alpha\ \beta c}$ (or $\Gamma^{\delta}_{\alpha c\ \beta}$) is equal to zero, both the indices α and β are also equal to zero and $\Gamma^{\delta}_{\alpha\ \beta c}$ ($\Gamma^{\delta}_{\alpha c\ \beta}$) coincides with $w\Gamma^{a}_{a\ c}$ ($W\Gamma^{a}_{ca}$).*

Theorem 1.4 *The following relation holds good*

$$w\Gamma^{\alpha a}_{\beta a\,\gamma c}=\Gamma^{\alpha}_{\beta\,\gamma c} \textit{ and } w\Gamma^{\alpha a}_{\beta b\,\gamma a}=\Gamma^{\alpha}_{\beta b\,\gamma} \quad .$$

Now we shall suppose such a continuos family of curves that one and only one curve belonging to it passes through every one point in the considered domain of the space. In order to define the Jacobian excovariant differential of a Jacobian extensor field defined on each curve belonging to it, for example, let us consider a Jacobian extensor $T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f}$ of the type indicated (α and β being G_1, and G_2; δ, γ being G_3 and G_4;) then the quantities given by

$$\begin{aligned}(1.5)\qquad \bar{\delta}\, T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f} = {} & dT^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f} + \sum_{\theta,\sigma}^{G_1} \Gamma^{\alpha}_{\theta\,\sigma h}\, T^{\theta}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f}\, dx^{\sigma h} \\ & - \sum_{\theta,\sigma}^{G_2} \Gamma^{\theta}_{\beta\,\sigma h}\, T^{\alpha}{}_{\theta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f}\, dx^{\sigma h} + \sum_{\theta,\sigma}^{G_3} \Gamma^{\delta d}_{\theta k\,\sigma h}\, T^{\alpha}{}_{\beta}{}^{\theta k}{}_{\gamma c}{}^{e}{}_{f}\, dx^{\sigma h} \\ & - \sum_{\theta,\sigma}^{G_4} \Gamma^{\theta k}_{\gamma c\,\sigma h}\, T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\theta k}{}^{e}{}_{f}\, dx^{\sigma h} + \sum_{h} \Gamma^{e}_{a\,h}\, T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{a}{}_{f}\, dx^{h} \\ & - \sum_{h} \Gamma^{a}_{f\,h}\, T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{a}\, dx^{h}\end{aligned}$$

are the components of a Jacobian extensor of the same kind as the original extensor $T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f}$, where the displacement $dx^{\delta c}$ means difference of the line elements at any two infinitesimally near points lying on any two infinitesimally near curves belonging to the family respectively. Such the quantties $\bar{\delta}\, T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f}$ and the quatities $T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f,\,\sigma h}$ defined by

$$\begin{aligned}(1.6)\qquad T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f,\,\sigma h} = {} & T^{\cdot}{}_{\cdot}{}^{\cdot}{}_{\cdot}{}^{\cdot}{}_{\cdot\,;\,(\sigma)h} + \sum \Gamma^{\alpha}_{\theta\,\sigma h}\, T^{\theta}{}_{\cdot}{}^{\cdot}{}_{\cdot}{}^{\cdot}{}_{\cdot} \\ & - \sum \Gamma^{\theta}_{\beta\,\sigma h}\, T^{\cdot}{}_{\theta}{}^{\cdot}{}_{\cdot}{}^{\cdot}{}_{\cdot} + \sum \Gamma^{\delta d}_{\theta k\,\sigma h}\, T^{\cdot}{}_{\cdot}{}^{\theta k}{}_{\cdot}{}^{\cdot}{}_{\cdot} \\ & - \sum \Gamma^{\theta k}_{\gamma c\,\sigma h}\, T^{\cdot}{}_{\cdot}{}^{\cdot}{}_{\theta k}{}^{\cdot}{}_{\cdot} + \sum \Gamma^{e}_{a\,h}\, T^{\cdot}{}_{\cdot}{}^{\cdot}{}_{\cdot}{}^{a}{}_{\cdot}\,\delta^{o}_{\sigma} \\ & - \sum \Gamma^{a}_{f\,h}\, T^{\cdot}{}_{\cdot}{}^{\cdot}{}_{\cdot}{}^{\cdot}{}_{a}\,\delta^{o}_{\sigma}\end{aligned}$$

are called the excovariant differential and the excovariant derivative of the Jacobian extensor T respectively and when $\bar{\delta}T=0$, we say that the J-extensor T is displaced parallelly. Hence we can get the following theorems:

Theorem 1.5 *The excovariant derivative of an Jacobian extensor* $T^{\alpha}{}_{\beta}{}^{\delta d}{}_{\gamma c}{}^{e}{}_{f}$: $T^{\alpha}{}_{\beta}{}^{\gamma d}{}_{\gamma c}{}^{e}{}_{f,\,\sigma h}$ *is an Jacobian extensor of the type indicated by the indices, of ranges with respect to* α, β, γ *and* δ *being* G_1, G_2, G_3 *and M respectively, and of functional order M.*

Theorem 1.6 *Let* V^{α} *be a Jacobian extensor, of ranges with respect to* α *being R, then the following relations are observed*

$$\bar{\delta}V^{o}=dV^{o}+w\Gamma^{a}_{a c}\,V^{o}\,dx^{c}\ (\equiv\delta V^{o})$$

for the tensor member of V^{α}, *and*

$$\bar{\delta}V^{\tilde{\alpha}}=dV^{\tilde{\alpha}}+\sum\Gamma^{\tilde{\alpha}}_{\beta\,\gamma c}\,V^{\beta}\,dx^{\gamma c}$$

for the member $V^{\tilde{\alpha}}$ ($\alpha=0, 1, \ldots .$ S and $S\leq R$) *of range S.*

Theorem 1.7 *If V is a scalar of weight* $-W$, *then we have*

$$(\bar{\delta}V)^{(\alpha)}=\bar{\delta}(V^{(\alpha)}) \quad .$$

These theorems are verified without difficulty by virtue of (1.1) and (1.3).

Corollary. *If a scalar* V^{o} *is displaced parallelly, that is,* $\delta V^{o}=0$, *then* $V^{(\alpha)}$ *is displaced parallelly too.*

Now, from the above argument, it should be noticed that the connection theory of Jacobian extensors by means of the excovariant differential contains the theory refferred to tensors, having arbitrary weight, in the space of the affine connection as a part.

At last of this paragraph we shall introduce the extended covariant derivaiive for the crossed Jacobian extensor [6], by use of the conectiion parameter $\Gamma^{\alpha a}_{\beta b\ \gamma c}$ and $Q^{\alpha}_{\beta\ \gamma c}$.

Theorem 1.8 *If the quantities* $T^{|\alpha|a}_{|\beta|b}$ *is the crossed Jacobian extensor of the type indicated by indices, of ranges with respect to* α *and* β *being* G_1 *and* G_2 *respectively, then the quantities* $\delta^* T^{|\alpha|a}_{|\beta|b}$ *and* $T^{|\alpha|a}_{|\beta|b,\ \gamma c}$ *deffned by the following equations*

$$\overline{\delta}^* T^{|\alpha|a}_{|\beta|b} = dT^{|\alpha|a}_{|\beta|b} + \sum_{\gamma,\delta} (\Gamma^{\alpha a}_{\delta d\ \gamma c}\, T^{|\delta|d}_{|\beta|b} + Q^{\alpha}_{\delta\ \gamma c} T^{|\delta|a}_{|\beta|b}) dx^{\gamma c} - \sum_{\gamma,\delta} (\Gamma^{\delta d}_{\beta b\ \gamma c}\, T^{|\alpha|a}_{|\delta|d} + Q^{\delta}_{\beta\ \gamma c} T^{|\alpha|a}_{|\delta|a}) dx^{\gamma c} \tag{1.7}$$

and

$$T^{|\alpha|a}_{|\beta|b,\ \gamma c} = T^{|\alpha|a}_{|\beta|b;\ \gamma c} + \sum_{\delta} (\Gamma^{\gamma d}_{\beta b\ \gamma c}\, T^{|\alpha|a}_{|\delta|d} + Q^{\delta}_{\beta\ \gamma c}\, T^{|\alpha|a}_{|\delta|b}) - \sum_{\delta} (\Gamma^{\alpha d}_{\beta b\ \gamma c}\, T^{|\alpha|a}_{|\delta|d} + Q^{\delta}_{\beta\ \gamma c}\, T^{|\alpha|a}_{|\delta|b}) \tag{1.8}$$

are called the excovariant differential and the excovariant derivative of the crossed Jacobian extensor $T^{|\alpha|a}_{|\beta|b}$ *respectively.* Then when $\overline{\delta}^* T = 0$, we say that the crossed J-extensor is displaced parallely.

Theorem 1.9 *The excovariant derivative of a crossed J-extensor* $T^{|\alpha|a}_{|\beta|b} : T^{|\alpha|a}_{|\beta|b,\ \gamma c}$ *is a crossed Jacobian extensor of the type indicated by the indices, of ranges with respect to* α, β *and* γ *being* G_1, G_2 *and M respectively, and of functional order M.*

Further we have the following theorems without explanation.

Theorem 1.10 *Let* $V^{|\alpha|a}$ *be a crossed Jacobian extensor, of ranges with respect to* α *being R, then we have*

$$\overline{\delta}^* V^{|0|\alpha} = dV^{|0|a} + (\Gamma^{a}_{d\ c}\, V^{|0|d} + wQ_c\, V^{|0|a}) dx^c (\equiv \delta V^{|0|a}) \ .$$

Theorem 1.11 *If* V^a *is a vector of weight* $-W$, *then we obtain*

$$\overline{\delta}^* V^{a(\alpha)} = (\delta V^a)^{(\alpha)} \ .$$

Theorem 1.12 *If a crossed Jacobian extensor* $V^{|\alpha|a}$ *is defined by the relation*

$$V^{|\alpha|a} = \sum_{\theta=0}^{\alpha} \binom{\alpha}{\theta} V^{\alpha-\theta} E^{\theta a} \tag{1.9}$$

where $V^{\alpha-\theta}$ *is a Jacobian extensor and* $E^{\theta a}$ *an extensor then we have*

$$\overline{\delta}^* V^{|\alpha|a} = \sum_{\theta} \binom{\alpha}{\theta} \overline{\delta} V^{\alpha-\theta} E^{\theta a} + \sum_{\theta} \binom{\alpha}{\theta} V^{\alpha-\theta} \delta^* E^{\theta\alpha} \ .$$

Proof. By means of (1.7) and (1.9) we have

$$\begin{aligned}
\overline{\delta}^* V^{|\alpha|a} &= d\{\sum_{\theta=0}^{\alpha} \binom{\alpha}{\theta} V^{\alpha-\theta} E^{\theta a}\} + \sum_{\delta=0}^{\alpha} \sum_{\gamma=0}^{\alpha-\delta} \{\Gamma^{\alpha a}_{\delta d\ \gamma c}\, V^{|\delta|d}\, dx^{\gamma\delta} + Q^{\alpha}_{\delta\ \gamma c}\, V^{|\delta|\alpha}\} dx^{\gamma c} \\
&= \sum_{\theta=0}^{\alpha} \binom{\alpha}{\theta} dV^{\alpha-\theta} E^{\theta a} + \sum_{\delta=0}^{\alpha} \sum_{\gamma=0}^{\alpha-\delta} \sum_{\theta-0}^{\delta} \binom{\delta}{\theta} Q^{\alpha}_{\delta\ \gamma c}\, V^{\delta-\theta} E^{\theta a}\, dx^{\gamma c} \\
&+ \sum_{\theta=0}^{\alpha} \binom{\alpha}{\theta} V^{\alpha-\theta}\, dE^{\theta a} + \sum_{\delta=0}^{\alpha} \sum_{\gamma=0}^{\alpha-\delta} \sum_{\theta=0}^{\delta} \binom{\delta}{\theta} \Gamma^{\alpha a}_{\delta d\ \gamma c}\, V^{\delta-\theta} E^{\theta a}\, dx^{\gamma c} \\
&= \sum_{\theta=0}^{\alpha} \binom{\alpha}{\theta} \{dV^{\alpha-\theta} + \sum_{\gamma} \sum_{\delta} \binom{\alpha-\theta}{\delta-\theta\ \gamma} (w\Gamma^{a}_{a\ c})^{(\alpha-\delta-\gamma)}\ V^{\cdot-\theta}\, dx^{\gamma c}\} E^{\theta a} \\
&+ \sum_{\theta=0}^{\alpha} \binom{\alpha}{\theta} V^{\alpha-\theta} E^{0a} + \sum_{\delta=0}^{\alpha} \sum_{\gamma=0}^{\alpha-\delta} \sum_{\varphi=\alpha-\delta}^{\alpha} \binom{\alpha}{\delta\ \gamma} \Gamma^{a\ (\alpha-\delta-\gamma)}_{d\ c}\ V^{\alpha-\varphi} E^{\varphi+\delta-\alpha\ d}\, dx^{\gamma c}
\end{aligned}$$

and using the fact that $\sum_{\delta=0}^{\alpha}\sum_{\gamma=0}^{\alpha-\delta}\sum_{\varphi=\alpha-\delta}^{\alpha}=\sum_{\varphi=0}^{\alpha}\sum_{\gamma=0}^{\varphi}\sum_{\delta=\alpha-\varphi}^{\alpha-\gamma}$ and putting $\Psi=\varphi+\delta-\alpha$, the last term in the right hand member reduces to

$$\sum_{\varphi=0}^{\alpha}\binom{\alpha}{\varphi} V^{\alpha-\varphi}\{dE^{\varphi a}+\sum\sum\binom{\varphi}{\gamma\psi}\Gamma^{a\ (\theta-\gamma-\psi)}_{dc} E^{\psi d}\, dx^{\gamma c}\}$$

Hence we have

$$\bar{\delta}^* V_{|\alpha|a}=\sum_{\theta=0}^{\alpha}\binom{\alpha}{\theta}\{\bar{\delta} V^{\alpha-\theta} E^{\theta a}+V^{\alpha-\theta}\,\delta^* E^{\theta a}\}\ .$$

In the same manner we have the following.

Theorem 1.13 *If a crossed Jacobian extensor* $V_{|\alpha|a}$ *is defined by the following:*

$$V_{|\alpha|a}=\sum_{\beta=\alpha}^{M}\binom{\beta}{\alpha}\binom{M}{\beta-\alpha}^{-1} V_\beta E_{M-\beta+\alpha\, a}$$

where the quantities V_β *is a Jacobian extensor and* $E_{\alpha a}$ *an extensor, then we have*

$$\bar{\delta}^* V_{|\alpha|a}=\sum_{\beta=\alpha}^{M}\binom{\beta}{\alpha}\binom{M}{\beta-\alpha}^{-1}\{\bar{\delta} V_\beta E_{M-\beta+\alpha\, a}+V_\beta\cdot\delta^* E_{M-\beta+\alpha,\, a}\}\ .$$

From the theorem 1.13 we have

Corollary. *If a scalar V of weight* $-W$ *is displaced parallelly and a vector* V^a *parallelly, then the crrossd Jacobian extensor* $V_{|\alpha|a}$ *defined by* $\sum_\theta\binom{\alpha}{\theta} V^{(\alpha-\theta)} V^{a(\theta)}$ *is displaced parallelly too.*

§ 2. *The Jacobian fundamental extensor of a Riemannian space.*

Let us suppose a one-parameter system of the metric tensor $g_{ab}(t)$ along a parameterized arc of class M: $x^a=x^a(t)$ in a n-dimensional Riemanian space, and construct the quantities

(2.1) $\qquad \overset{M}{g}_{\alpha\beta}\equiv\binom{M}{\alpha\beta} g^{w(M-\alpha-\beta)}$ where $g=|g_{ab}(t)|$,

then the quantities $\overset{M}{g}_{\alpha\beta}$ are the components of a Jacobian extensor of the type indicated. we shall call such the quantities $\overset{M}{g}_{\alpha\beta}$ the components of the fundamental Jacobian extensor in the Riemannian space of M-th order line-elements: $R_n^{(M)}$. The fundamental Jacobian extensor $\overset{M}{g}_{\alpha\beta}$ is symmetric with respect to α, β and $|\overset{M}{g}_{\alpha\beta}|$ not zero. Hence we can depne the unique reciprocal Jacobian excontravariant fundamental extensor $\underset{M}{g}_{\alpha\beta}$ that satisfines the following equation:

(2.2) $\qquad \overset{M}{g}_{\alpha\beta}\,\underset{M}{g}^{\alpha\beta}=\delta^\gamma_\beta$

where symbol δ denotes Kronecker delta. Then the quantities $\underset{M}{g}^{\alpha\beta}$ is given by the following:

$$\underset{M}{g}^{\alpha\beta}=\left[\begin{smallmatrix}\alpha\ \beta\\ M\end{smallmatrix}\right] g^{-w(\alpha+\beta-M)}$$

We define the quantities $V_\beta=\overset{M}{g}_{\alpha\beta} V^\alpha$ determined by components of a Jacobian extensor as the Jacobian excovariant componets of the Jacobian extensor, then we obtaine $V_\beta=\underset{M}{g}^{\alpha\beta} V_\beta$. Then we have

Theorem 2.1 *If* f^* *is a scalar of weight* $+W$ *derived from the equation* $f^*=g^w f$, *where f is a scalar of weigh* $-W$, *then we have*

$$\overset{M}{g}_{\alpha\beta} f^{(\alpha)} = \binom{M}{\beta} f^{*(M-\beta)} \quad and \quad \underset{M}{g}^{\alpha\beta} \binom{M}{\beta} f^{*(M-\beta)} = f^{(\alpha)}$$

proof. Differentiating the equation: $f^* = G^w f$ $M-\beta$ times by Leipnitz rule and multiplying the binomial coefficient $\binom{M}{\beta}$ to the both members of the resulting equation, then the theorem is established.

Let V^α be the components of an Jacobian excontravariant extensor in $R_n^{(M)}$, then $\overset{M}{g}_{\alpha\beta} V^\alpha V^\beta$ is a scalar. Hence we shaa call $(\overset{M}{g}_{\alpha\beta} V^\alpha V^\beta)^{\frac{1}{2}}$ the length of the Jacobian excontravariant extensor V^α. By means of this definition we obtaine.

Theorem 2.2 *If V is the length of a Jacobian excontravariant extensor V^α, then we get $V^2 = \underset{M}{g}^{\alpha\beta} V_\alpha V_\beta$.*

Theorem 2.3 *When V is the absolute scalar derived fiom the scalar of weight $-W$ by the relation: $V^2 = g^w f^2$, we have*

$$\frac{\overset{M}{d} V^2}{dt^M} = \overset{M}{g}_{\alpha\beta} f^{(\alpha)} f^{(\beta)}.$$

§ 3. Jacobian Christoflel symbol in the Riemannian space.

We shall call the Jacobian Christoffol symbole of the first and of the second kind, the quantities $\overset{M}{\Gamma}_{\alpha\beta,\gamma c}$ and $\overset{M}{\Gamma}{}^\alpha_{\beta\gamma c}$ defined by

$$\overset{M}{\Gamma}_{\alpha\beta,\gamma c} = \frac{1}{2}(\overset{M}{g}_{\gamma\beta;\alpha c} + \overset{M}{g}_{\alpha\gamma;\beta c} - \overset{M}{g}_{\alpha\beta;\gamma c}) \tag{3.1}$$

and

$$\Gamma^\alpha_{\beta\gamma c} = \sum_{\delta=0}^{M} \underset{M}{g}^{\alpha\delta} \overset{M}{\Gamma}_{\beta\gamma;\delta c} \tag{3.2}$$

respectively. From this definition we have the following:

Theorem 3.1 *Jacobian Christoffel symbol of the second kind $\overset{M}{\Gamma}{}^\alpha_{\beta\gamma c}$ coincides with the Jacobian conection parameter: $\binom{\alpha}{\beta\gamma}(w\Gamma^a_{ac})^{(\alpha-\beta-\gamma)}$, where Γ^a_{bc} indicates Christoffel symbol of the second kind by means of g_{ab} and g^{ac}.*

Proof. The first term in the right-hand member of (3.2) reduce

$$\sum_{\delta=0}^{M} g^{\alpha\delta} \overset{M}{g}_{\delta\gamma;\beta c} = \sum_{\delta=0}^{M} \left[\begin{smallmatrix}\alpha\,\beta\\ M\end{smallmatrix}\right] g^{-w(\alpha+\gamma-M)} \binom{M}{\delta\,\gamma} g^{w(M-\delta-\gamma)}{}_{;\beta c}$$

$$= \binom{\alpha}{\beta\gamma}\left(\frac{1}{2} g^{-w} g^w_{;c}\right)^{(\alpha-\beta-\gamma)}$$

$$= \binom{\alpha}{\beta\gamma}\left(\frac{1}{2}\frac{\partial \log g^w}{\partial x^c}\right)^{(\alpha-\beta-\gamma)}.$$

Similarly

$$\sum_{\delta=0}^{M} g^{\alpha\delta} \overset{M}{g}_{\beta\delta;\gamma c} = \binom{\alpha}{\gamma\beta}\left(\frac{1}{2}\frac{\partial \log g^w}{\partial x^c}\right)^{(\alpha-\gamma-\beta)},$$

$$\sum_{\delta=0}^{M} g^{\alpha\delta} \overset{M}{g}_{\beta\alpha;\delta c} = \binom{\alpha}{\beta\gamma}\left(\frac{1}{2}\frac{\partial \log g^w}{\partial x^c}\right)^{(\alpha-\gamma-\beta)}.$$

Consequently we have

$$\overset{M}{\Gamma}{}^{\alpha}_{\beta\gamma c}=\binom{\alpha}{\beta\alpha}(w\Gamma^{\alpha}_{\alpha c})^{(\alpha-\beta-\gamma)}$$

Using the same manner in the Y. Katurada's paper, we have the following theorems:

Theorem 3.2 *If* $\overset{R}{\Gamma}_{\alpha\beta,\gamma c}$ *is Jacobian Christoffel symbol of the first kind by means of* $\overset{R}{g}_{\alpha\beta}$, *where* $\overset{R}{g}_{\alpha\beta}=\binom{R}{\alpha\beta}g^{w(R-\alpha-\beta)}$ *then we see that*

$$\binom{M+\sigma}{R}\overset{R}{\Gamma}_{\alpha\beta,\sigma c}=\binom{M+\sigma}{M}\overset{M}{\Gamma}_{\alpha\beta,\,M-R+\sigma c},$$

$$(\alpha,\ \beta,\ \sigma=0,\cdots,\ R,\ and\ R\leqq M).$$

Theorem 3.3 *There exists the relation*

$$\overset{R}{\Gamma}{}^{\gamma}_{\alpha\,\beta c}=\sum_{\sigma=M-R}^{M}g^{\gamma\sigma}\overset{M}{\Gamma}_{\alpha\beta,\,\rho c},\qquad (\alpha,\ \beta,\ \gamma=0,\ \cdots,\ R),$$

$\overset{R}{\Gamma}{}^{\gamma}_{\alpha bc}$ *being Jacobian Christoffel symbol by means of* $\overset{R}{g}_{\alpha\beta}$ *and* $\underset{R}{g}^{\alpha\gamma}$.

Corollary. *If* $\overset{o}{\Gamma}{}^{a}_{bc}$ *is Christoffel symbol by means of the metric tensor* g_{ab}, *then*

$$w\overset{o}{\Gamma}{}^{a}_{bc}=\overset{M}{g}{}^{\gamma M}\overset{M}{\Gamma}_{\alpha\beta,\,Mc}\quad \alpha,\ \beta,\ \gamma=0\qquad (M:\ \textit{not summing}).$$

Hence, from the theorem 3.3, we know that the quantities $\sum_{\sigma=M-R}^{M}g^{\gamma\sigma}\overset{M}{\Gamma}_{\alpha\beta,\sigma c}$ being a part of Jacobian Christoffel symbol by means of $\overset{M}{g}_{\alpha\beta}$ and $\underset{M}{g}^{\alpha\gamma}$ may be adopted as Jacobian connection parameter for a member of range R of a Jacobian extensor and especially $\underset{M}{g}^{\gamma c Me}\overset{M}{\Gamma}_{a b,Me}$ (M: not summing) as that for a tensor member having weight. Such a set of (M+I) quantities in $R_n^{(M)}$, i.e., $\sum_{\sigma=M-R}^{M}g^{\gamma\sigma}\overset{M}{\Gamma}_{\alpha\beta,\sigma c}(=\overset{R}{\Gamma}{}^{\gamma}_{\alpha\,\beta c})$, $R=0,\ \cdots,\ M$ is called Jacobian Christoffel symbol.

We shall adopt the definition of excovariant differental of a Jacobian extensor given by (1.1) in § 1. Then we have the following theorems:

Theorem *3.4* *The excovariant differential of the Jacobian extensor* $\overset{R}{g}_{\alpha\beta}$, $\underset{R}{g}^{\alpha\beta}$ (R=0, ⋯, M) *and* δ^{α}_{β} *is zero respectively.*

Theorem 3.5 *If* $T^{\alpha_1\cdots\alpha_A}{}_{\beta_1\cdots\beta_b}$ *is a Jacobian extensor of range R, its excovariant differential* $\bar{\delta}^{R}\,T^{\alpha_1\cdots\alpha_\beta}{}_{\beta_1\cdots\beta_b}$ *by means of* $\overset{R}{\Gamma}{}^{\alpha}_{\beta\,\gamma c}$ *is equal to* $\bar{\delta}\,T^{\alpha_1\cdots\alpha_A}{}_{\beta_1\cdots\beta_b}$.

Theorem 3.6 *If* V^{α} *is a Jacobian extensor of range R satisfying* $\bar{\delta}\,V^{\alpha}=0$, *that is, the Jacobian extensor is displaced parallely to themselve, then we have* $\delta^{*}V=0$ *where V mean the length* V^{α}.

Theorem 3.7 *If V is the scalar derived from the relation* $V^2=g^{w}f^2$, *where f is a scalar of weigh* $-W$, *and if* $\delta f=0$, *then the Jacobian extensor* $f^{(\alpha)}$ *is displaced parallelly and the scalar* $\dfrac{d^{\alpha}V^2}{dt^{\alpha}}$ ($\alpha=0,\ \cdots,$ R) *does not change.*

At last let us consider an Euclidean space, then we have $g=1$. Hence Jacobian fundamental extensor $\overset{M}{g}_{\alpha\beta}$ and $\underset{M}{g}^{\alpha\beta}$ become equal to $\overset{M}{\delta}_{\alpha\beta}$ and $\underset{M}{\delta}^{\alpha\beta}$ respectively, where $\overset{M}{\delta}_{\alpha\beta}=\binom{M}{\alpha}\delta_{M-\alpha,\beta}$ and $\underset{M}{\delta}^{\alpha\beta}=\binom{M}{\alpha}^{-1}\delta^{M-\alpha,\beta}$. Evidently, we have the relation $\sum_{\alpha=0}^{M}\overset{M}{\delta}_{\alpha\beta}\underset{M}{\delta}^{\alpha\gamma}=\delta^{\gamma}_{\beta}$. Further, in the cartesian coordinate system, the Jacobian Christoffel symbol $\overset{R}{\Gamma}{}^{\alpha}_{\beta\gamma c}$ vanishes identically and the excovariant differential becomes the ordinary differential.

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