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Infinitesimal Transformations in the Cartan Space

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矢野 晋平：Cartan 空間の無限小變換

Introduction

The geometrical property of continuous groups of transformations in the Riemannian space, is one of the most interestings in the theory of groups[1]. K.YANO[2] and E.T.DAVIES[3] extended this property by making use of the *Lie* derivatives in to generalized spaces. On the other hand, in investigation of the transformations in the *Finsler* space, M.S.KNEBELMAN[4] and E.T.DAVIES[5] formulated many interesting theories which E.T.DAVIES[6] establisched in the metric space.

In the present paper we show whether analogous theories exist or not in the *Cartan* space.[7]

§ 1. Fundamental equations in the n-dimensional Cartan space.

In an n-dimensional manifold, if any $(n-1)$-dimensional sub-space is given by the equations

(1.1) $$x^i = x^i(y^1,\cdots,y^{n-1}) \qquad (i=1,\cdots,n),$$

and if the $(n-1)$-dimensional area of a region R_{n-1} in the sub-space, is given by the $(n-1)$-ple integral

(1.2) $$O = \int_{n-1} F(x,\partial x/\partial y)dy^1,\cdots,dy^{n-1},$$

then the n-dimensional manifold is called the *Cartan* space[8].

In this space we define the elements of hypersurface u_α by the following determinants made of (1.1)

(1.3) $$u_\alpha = (-1)^{\alpha+1} \begin{vmatrix} \frac{\partial x^1}{\partial y^1} \cdots \frac{\partial x^{\alpha-1}}{\partial y^1} & \frac{\partial x^{\alpha+1}}{\partial y^1} \cdots \frac{\partial x^n}{\partial y^1} \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ \frac{\partial x^1}{\partial y^{n-1}} \cdots \frac{\partial x^{\alpha-1}}{\partial y^{n-1}} & \frac{\partial x^{\alpha+1}}{\partial y^{n-1}} \cdots \frac{\partial x^n}{\partial y^{n-1}} \end{vmatrix}.$$

Let $T^i{}_j\ (x, u)$ be the components of a mixed tensor of degree zero in the u, then the two kinds of differential operators[9] $|_k$ and $|^k$ are applied to the $T^i{}_j$ in the following expressions

1) See, [7]
2) See, [10], §11
3) See, [4]
4) See, [8]
5) See, [5]
6) See, [6]
7) The notations in our paper, are the same as those used by L.BERWALD [1], and K.YANO [9], omitting * in $\Gamma^*{}_i{}^j{}_k$.
8) See, [1] Einleitung, see [2] §6
9) See, [1] §8, [9] §8

(1.4) $$T^i{}_j|_k = T^i{}_{j,k} + T^i{}_j\|^\beta \Gamma_{\beta 0 k} + T^\beta{}_j \Gamma_\beta{}^i{}_k \quad T^i{}_\beta \Gamma_j{}^\beta{}_k,$$

(1.5) $$T^i{}_j|^k = T^i{}_j\|^k + T^\beta{}_j A_\beta{}^{ik} - T^i{}_\beta A_j{}^{\beta k},$$

where

(1.6) $$l_i = (\sqrt{g}/L) u_i,$$

(1.7) $$T^i{}_j\|^k = (L/\sqrt{g}) T^i{}_j;^k,$$

Throughout the paper we shall use the notations, $T^i{}_{j,k} = \partial T^i{}_j / \partial X^k$, $T^i{}_j;^k = \partial T^i{}_j / \partial u_k$.

In the Cartan space there are four kinds of curvature tensors[1)] given by the following expressions

(1.8) $$K_i{}^j{}_{kh} = 2(\Gamma_i{}^j{}_{[k,h]} + \Gamma_i{}^\alpha{}_{[k} \Gamma_{h]}{}^j{}_\alpha + \Gamma_i{}^j{}_{[k}\|^\alpha \Gamma_{h]0\alpha}),$$

(1.9) $$R_i{}^j{}_{kh} = K_i{}^j{}_{kh} - A_i{}^{j\alpha} K_{\alpha 0 kh},$$

(1.10) $$P_i{}^j{}_k{}^h = \Gamma_i{}^j{}_k\|^h - A_i{}^{jh}|_k - A_i{}^{j\alpha} l_\beta \Gamma_\alpha{}^\beta{}_k|^h,$$

(1.11) $$S_i{}^{jkh} = 2 A^{\alpha j[k} A_{\alpha i}{}^{h]}.$$

§2 The relations of the differential operators to the curvature tensors.

On consideration of (1.7), from (1.4), (1.5), (1.8) and (1.9) we have

(2.1) $$2 T^i{}_j|_{[hk]} = T^i{}_j\|^\alpha K_{\alpha 0 hk} + T^\alpha{}_j K_\alpha{}^i{}_{hk} - T^i{}_\alpha K_j{}^\alpha{}_{hk} = T^i{}_j|^\alpha R_{\alpha 0 hk} + T^\alpha{}_j R_\alpha{}^i{}_{hk} - T^i{}_\alpha R_j{}^\alpha{}_{hk}.$$

By virtue of the relation[2)] $(L/\sqrt{g})\|^i = L/\sqrt{g}\,(l^i - A^i)$ we see that

(2.2) $$T^i{}_j\|^{[kh]} = T^i{}_j|^{[k}(l^{h]} - A^{h]}).$$

Put the metric tensors g_{ij} and g^{ij} in to these equations and in consequence of the relations[3)] $g^{ij}\|^h = -2A^{ijh}$ and $g_{ij}\|^h = 2A_{ij}{}^h$ we have

(2.3) $$A_i{}^{j[k}\|^{h]} = 2A^{\alpha j[k} A_{i\alpha}{}^{h]} + A_i{}^{j[k}(l^{h]} - A^{h]}).$$

Since $\Gamma_j{}^i{}_k\|^h$ is a tensor of order four[4)], from the connection parameters $\Gamma_j{}^i{}_k$ we have

(2.4) $$\Gamma_j{}^i{}_m\|^{[hk]} = \Gamma_j{}^i{}_m\|^{[h}(l^{k]} - A^{k]}).$$

If we substitute the right side of (1.6) in the relations[5)]

(2.5) $$l_{i,j} + l_i\|^\alpha \Gamma_{\alpha 0 i} - \Gamma_{i0j} = l_{i|j} = 0,$$

in use of the equations[6)]

(2.6) $$l_i\|^\alpha = \delta^\alpha{}_i - l_i(l^\alpha - A^\alpha), \qquad l^i\|^\alpha = g^{i\alpha} - l^i(l^\alpha + A^\alpha),$$

we have $(L/\sqrt{g})_{,h} + (l^\alpha - A^\alpha)\Gamma_\alpha{}^\beta{}_h u_\beta = 0$. From these equations and (2.2) we obtain finally

(2.7) $$T^i{}_j|_h\|^k - T^i{}_j\|^k|_h = T^i{}_j\|^\alpha \Gamma_\alpha{}^\beta{}_h\|^k l_\beta + T^\alpha{}_j \Gamma_\alpha{}^i{}_h\|^k - T^i{}_\alpha \Gamma_j{}^\alpha{}_h\|^k.$$

Since the necessary and sufficient condition for that a Cartan space become an affine connected space is that $\Gamma_\alpha{}^i{}_j\|^k = 0$[7)], from (2.7) we get

Theorem [2.1]. *If the Cartan space is an affine connected space, then two operators* $|_h$ *and* $\|^k$ *may be interchangeable.*

By virtue of (2.2) and (1.5) we have

(2.8) $$T^i{}_j|^k\|^h - T^i{}_j\|^h|^k = 2T^i{}_j|^{[k}(l^{h]} - A^{h]}) - T^i{}_j|^\alpha A_\alpha{}^{hk} + T^\alpha{}_j A_\alpha{}^{ik}\|^h - T^i{}_\alpha A_j{}^{\alpha k}\|^h.$$

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1) See [1], §12 and [9], §9.
2) See [1], §5
3) See [1], §5
4) See [1], §16
5) See [1], §8
6) See [1], §5
7) See [9], §12

Since $T^l{}_{j|h}$ and $T^l{}_j|^k$ are the tensors, from (2.7) and (1.10) we have

(2.9) $$T^l{}_{j|h}|^k - T^l{}_j|^k{}_{|h} = T^l{}_j|^\alpha T_\alpha{}^\beta{}_h\|^k l_\beta - T^l{}_{j|\alpha} A_h{}^{\alpha k} + T^\alpha{}_j P_\alpha{}^l{}_h{}^k - T^l{}_\alpha P_j{}^\alpha{}_h{}^k.$$

Considering (1.5), (2.2), (2.3) and (1.11) we find

(2.10) $$2T^l{}_j|^{[k}\|^{h]} = 2T^l{}_j|^{[k}(l^{h]} - A^{h]}) + T^\alpha{}_j S_\alpha{}^{lkh} - T^l{}_\alpha S_j{}^{\alpha kh} + 2T^\alpha{}_j|^{[h} A_\alpha{}^{l|k]} - 2T^l{}_\alpha|^{[h} A_j{}^{\alpha|k]}.$$

From these equations we have

(2.11) $$2T^l{}_j|^{[kh]} = 2T^l{}_j|^{[k}(l^{h]} - A^{h]}) + 2T^l{}_j|^\alpha A_\alpha{}^{[kh]} + T^\alpha{}_j S_\alpha{}^{lkh} - T^l{}_\alpha S_j{}^{\alpha kh}.$$

Taking l_i in place of $T^l{}_j$ in the above equations and considering the relations $S_{i0}{}^{kh} = 0$[1] and $l_i|^j = (\delta^j{}_i - l_i l^j)$[2], we find $A_i{}^{[jk]} = \delta^{[j}{}_i A^{k]}$. From these equations we have

Theorem [2.2]. *If the condition* $A^i = 0$ *is satisfied, then the tensor* A^{ijk} *is symmetric in the indices* i, j, k.

§ 3 The Lie derivatives

Let us consider a one-parameter continuous group of transformations defined by finite equations $\bar{x}^i = \bar{x}^i(x.a)$, and the infinitesimal transformations of the group given by

(3.1) $$\bar{x}^i = x^i + \xi^i(x)\delta t,$$

where δt is an infinitesimal constant and ξ^i a contravariant vector field depending on a point.

If we introduce the notations $\partial x^i/\partial y^a = x^i{}_a$ and $\delta x^i{}_a = \bar{x}^i{}_a - x^i{}_a = \xi^i{}_{,j} x^j{}_a \delta t$, from (1.3) we have

(3.2) $$\sum_{a=1}^{n-1} (\partial u_i/\partial x^j{}_a) x^k{}_a = u_i \delta^k{}_j - u_j \delta^k{}_i.$$

Put the elements of hypersurface deformed by the transformation (3.1) in the form $\bar{u}_i = u_i + \delta u_i$, where $\delta u_i = (\partial u_i/\partial x^j{}_a)\delta x^j{}_a$. On neglecting terms of higher orders, considering (3.2) we have

(3.3) $$\delta u_i = (u_i \delta^k{}_j - u_j \delta^k{}_i)\xi^j{}_{,k}\delta t.$$ [3]

If we define, with reference to the transformation (3.1), the *Lie* derivative of any geometric object $T{:}{:}$ (e.g. a tensor, tensor density, connexion parameters, etc.) as

$$\mathrm{X}T{:}{:} = \lim_{\delta t \to 0} \frac{T{:}{:}(\bar{x},\bar{u}) - \bar{T}{:}{:}(\bar{x},\bar{u})}{\delta t}.$$

In consequence of the relations $\partial \bar{x}^i/\partial x^j = \delta^i{}_j + \xi^i{}_{,j}\delta t$ and $\partial x^j/\partial \bar{x}^i = \delta^j{}_i - \xi^j{}_{,i}\delta t$, from (3.3) the Lie derivative of tensor $T^i{}_j$ can be written in the form[4],

(3.4) $$XT^i{}_j = T^i{}_{j|\alpha}\xi^\alpha - T^i{}_j\|^\alpha \Phi_{|\alpha} + T^i{}_\alpha \xi^\alpha{}_{|j} - T^\alpha{}_j \xi^i{}_{|\alpha}, \quad \text{where } \Phi = \xi^i l_i.$$

If $T^i{}_j(x,u)$ are the components of a tensor density of weight p, considering the equations $|\partial x/\partial \bar{x}|^p = 1 - p\xi^\alpha{}_{,\alpha}\delta t$, we have

(3.5) $$\mathrm{X}T^i{}_j = T^i{}_{j,\alpha}\xi^\alpha - T^i{}_j\|^\alpha \Phi_{|\alpha} + pT^i{}_j \xi^\beta{}_{|\beta} - T^\alpha{}_j \xi^i{}_{|\alpha} + T^i{}_\alpha \xi^\alpha{}_{|j}.$$

Let $\mathfrak{T}^i{}_j$ be the components of a tensor of weight p and homogeneous of degree q in the u, then we have

(3.6) $$X\mathfrak{T}^i{}_j = \mathfrak{T}^i{}_{j,\alpha}\xi^\alpha - \mathfrak{T}^i{}_j\|^\alpha \xi^\beta{}_{,\alpha} l_\beta + \mathfrak{T}^\alpha{}_j \xi^i{}_{,\alpha} + \mathfrak{T}^i{}_\alpha \xi^\alpha{}_{,j} + (p+q)\mathfrak{T}^i{}_j \xi^\alpha{}_{,\alpha}.$$

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1) See [1] §12
2) See [9] §8
3) See [6] §2
4) See [6] §2

Since u_i is a covariant vector density of weight -1 and dx^i a contravariant vector. From (3.4) and (3.5) we get $Xu_i=0$, $Xdx^i=0$, consequently we have

Theorem [3.1]. *The Lie derivatives of the hypersurface and line elements are equal to zero.*

Similarly, we have

Theorem [3.2]. *The Lie derivatives of ξ^i and $\delta^i{}_j$ are equal to zero.*

If we substitute l_i in (3.4) and considering (2.5) and (2.6) we have $Xl_i=l_i\,(l^\alpha-A^\alpha)\Phi_{|\alpha}$. If these equations be contracted by l^i, we have

$$Xl_i=l_i l^\alpha X l_\alpha. \tag{3.7}$$

Substitute the right side of (1.6) in (3.7), from theorem [3.1] we have

$$(L/\sqrt{g})\,X(\sqrt{g}/L)=l^\alpha X l_\alpha. \tag{3.8}$$

From the parameters $\Gamma_j{}^i{}_k$, considering the equations $\xi^i{}_{|j}=\xi^i{}_{,j}+\Gamma_\alpha{}^i{}_j\xi^\alpha$, we obtain

$$X\Gamma_j{}^i{}_k=\xi^i{}_{|jk}+\xi^\alpha K_j{}^i{}_{k\alpha}-\Gamma_k{}^i{}_j\|^\alpha\Phi_{|\alpha}.^{1)} \tag{3.9}$$

Since $\Gamma_k{}^i{}_j\|^\alpha$ is a tensor, we have

Theorem [3.3]. *The Lie derivative of connection parameter $\Gamma_j{}^i{}_k$ is a tensor.*

The application to the metric tensor g_{ij} will therefore give

$$Xg_{ij}=\xi_{i|j}+\xi_{j|i}-2A_{ij}{}^\alpha\Phi_{|\alpha}\,. \tag{3.10}$$

The *equations of Killing*, in the Riemannian geometry, are given by the relation

$$Xg_{ij}=0. \tag{3.11}$$

Therfore, we may say the transformation (3.1) giving (3.11) an *infinitesimal motion*. For the motion, considering (3.11), (3.7) and the relation $X(l^i\,l_i)=0$, we have

$$Xl_i=0. \tag{3.12}$$

§ 4 Alternation of Lie derivative and the differential operators.

The following alternate formula will be needed later.

From (2.1), (2.7) and (3.9) we obtain

$$X(T^i{}_{j,k})-(XT^i{}_j)_{,k}=T^i{}_j|^\alpha X\Gamma_\alpha{}^\beta{}_k l_\beta+T^\alpha{}_j X\Gamma_\alpha{}^i{}_k-T^i{}_\alpha X\Gamma_j{}^\alpha{}_k. \tag{4.1}$$

These equations imply

Theorem [4.1]. *If infinitesimal transformation* (3.1) *satisfy conditions $X\Gamma_j{}^i{}_k=0$, then the Lie derivative and the operator $|_k$ are interchangeable.*

If $\mathfrak{T}^i{}_j$ is a tensor density of weight p and homogeneous of degree q in the u, then $\mathfrak{T}^i{}_j{}^{;k}$ is a tensor of weight $p+1$ and degree $q-1$. Therefore considering (3.6) we have

$$X(\mathfrak{T}^i{}_j{}^{;k})-(X\,\mathfrak{T}^i{}_j)^{;k}=0. \tag{4.2}$$

Theorem [4.2]. *Let $\mathfrak{T}^i{}_j$ be a tensor of any weight and degree in the u, then the Lie derivative and partial differentiation with u are mutually interchangeable.*

From (3.8) and (4.2) we have

$$X(\mathfrak{T}^i{}_j\|^k)-(X\mathfrak{T}^i{}_j)\|^k=-\mathfrak{T}^i{}_j\|^k l^\alpha X l_\alpha. \tag{4.3}$$

from this equation and (1.7) we have

Theorem [4.3]. *If $\mathfrak{T}^i{}_j$ is independent of u, then $X\mathfrak{T}^i{}_j$ is same also.*

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1) See [6], §2

By virtue of (3.12) we obtain

Theorem [4.4]. *If* (3.1) *is a motion, the Lie derivative and the operator* $|^k$ *are interchangeable.*

Finally we have from (4.3)

(4.4) $$X(T^l{}_j|^k)-(XT^l{}_j)|^k=-T^l{}_j|^k l^\alpha Xl_\alpha+T^\alpha{}_j XA_\alpha{}^{lk}-T^l{}_\alpha XA_j{}^{\alpha k}.$$

Since $\Gamma_j{}^l{}_k;^h$ is a tensor density of weight 1 and of the homogeneous degree -1 in the u, then considering (4.2) and (4.3), we have

(4.5) $$X(\Gamma_j{}^l{}_k\|^h)-(X\Gamma_j{}^l{}_k)|^h=-\Gamma_j{}^l{}_k|^h l^\alpha Xl_\alpha.$$

From the relation $\bar{\Gamma}_j{}^l{}_k=\Gamma_j{}^l{}_k+X\Gamma_j{}^l{}_k\delta t,\quad K_j{}^l{}_{kh}=2\Gamma_j{}^l{}_{[k,h]}+2\Gamma_j{}^\alpha{}_{[k}\Gamma_{h]}{}^l{}_\alpha+2\Gamma_j{}^l{}_{[k}\|^\alpha\Gamma_{h]0\alpha}$, considering (4.5) and (3.7), we find

(4.6) $$XK_i{}^j{}_{kh}=2(X\Gamma_i{}^j{}_{[k})_{|h]}+2\Gamma_i{}^j{}_{[k}\|^\alpha X\Gamma_{h]}{}^\beta{}_\alpha l_\beta.$$

From these equations we have

Theorem [4.5]. *If the transformations* (3.1) *satisfy the relations* $X\Gamma_j{}^i{}_k=0$, *the Lie derivative of* $K_i{}^j{}_{kh}$ *vanishes.*

Substitute g_{ij} in (4.3), we have

(4.7) $$2XA_{ij}{}^k=-2A_{ij}{}^k l^\alpha Xl_\alpha+(Xg_{ij})|^k,$$

from these equations and (3.12) we have

Theorem [4.6]. *When the space admits an infinitesimal motion* (3.1), *the Lie derivative of connexion* $A_{ij}{}^k$ *vanishes.*

Put g_{ij} in (4.1), from the relation $g_{ij|k}=0$ we have

(4.8) $$(Xg_{ij})_{|k}=(g_{\beta j}\delta^\alpha{}_j+g_{i\beta}\delta^\alpha{}_j-2A_{ij}{}^\alpha l_\beta)X\Gamma_\alpha{}^\beta{}_k.$$ [1]

From these equations we have

Theorem [4.7]. *When the space admits an infinitesimal motion* (3.1), *the Lie derivative of connexion* $\Gamma_j{}^i{}_k$ *is equal to zero.*

§ 5 Groups of motions and affine motions

In this chapter, we shall study the continuous groups of motions, then from theorems [4.6], [4.7] and (3.12) we have

(5.1) $$XA_i{}^{jk}=0,\qquad X\Gamma_i{}^j{}_k=0,\qquad Xl_i=0.$$

By virtue of (3.11) and theorem [3.1] we see that $X(ds)=X(g_{ij}dx^i dx^j)=0$, hence we have

Theorem [5.1]. *When the space undergoes a motion into itself, lengths and angles remain unaltered.*

A necessary and sufficient condition that the *trajectories* of two motions be the same is that (3.9) be satisfied by ξ^i and $\bar{\xi}^i$, where $\bar{\xi}^i=\rho\xi^i$, then we have $\rho_{,\alpha}(2A_{ij}{}^\alpha\xi^\beta l_\beta-\delta^\alpha{}_i\xi_j-\delta^\alpha{}_j\xi_i)=0$, if the matrix of the coefficients of $\rho_{,\alpha}$ is of rank n, we obtain $\rho_{,\alpha}=0$, that is $\rho=constant$: we have

Theorem [5.2]. *Two groups of motions cannot have the same trajectories.*

If the space admits absolute parallelism, then denoting by $h^i{}_a$ the n contravariant vectors of an *absolutely parallel ennuple*,[2] the suffixes i and j being a component and a and b being a number of the vectors, we have

(5.2) $$h^i{}_a h_{bi}=\delta_{ab},\qquad \sum_{a=1}^{n}h^i{}_a h_{aj}=\delta^i{}_j.$$

1) See [6] §2
2) See [10] §3

From the relations $Dh^i{}_a = h^i{}_{a|j}dx^j + h^i{}_a|^j Dl_j = 0$, we have

(5.3) $h^i{}_{a;j} = 0, \qquad h^i{}_a{}^{|j} = 0, \qquad (i,j,a = 1,\cdots,n).$

Put $h^i{}_a$ in to (2.1) and considering (5.2) and (5.3), we have $R_i{}^j{}_{kh} = 0$ and $K_i{}^j{}_{kh} = 0$. Similarly, from (2.9) and (2.11) we have $P_i{}^j{}_k{}^h = 0$ and $S_i{}^{jkh} = 0$, consequently we have

Theorem [5.1]. *If the space admits absolute parallelism, all the curvature tensors vanish.*

From (5.2) and (5.3) we have

(5.4) $$\Gamma_i{}^j{}_k = \sum_{a=1}^{n} h^j{}_a h_{ai,k} + A_i{}^{j\beta}\Gamma_{\beta 0 k}, \qquad h_{ai}|^j = h_{ak}A_i{}^{kj}.$$

Let us multiply l_j in the first equation of (5.4) and contract on j, then $\Gamma_{\beta 0 k}$ can be solved, considering the relations $A_i{}^{j\beta}l_j = l_i A_\beta$ and $(\delta^\alpha{}_j - l_j A^\alpha)\delta^i{}_\alpha - l_\alpha A^i) = \delta^i{}_j$,

Putting the obtaind value of $\Gamma_{\beta 0 k}$ in to the equation again, we have

(5.5) $$\Gamma_i{}^j{}_k = \sum_{a=1}^{n} h^j{}_a h_{ai,k} + \sum_{a=1}^{n} A_i{}^{j\alpha} h^\beta{}_a l_\beta h_{a\alpha,k}$$

and from the second equation (5.4) we have

(5.6) $$A_i{}^{jk} = \sum_{a=1}^{n} h_{ai}|^k h^j{}_a.$$

Suppose that the space admits an infinitesimal transformation (3.1) satisfying the equations (5.1), from (5.3) and theorem [4.1] we have

(5.7) $(Xh^i{}_a)_j = 0.$

Since $Xh^i{}_a$ is a vector, there are certain scalars $C^b{}_a$ satisfing the relations

(5.8) $Xh^i{}_a = C^b{}_a h^i{}_b.$

From (5.7) and (5.3) we have $C^b{}_{a;j} = 0$, this condition may be written in the equation

(5.9) $C^b{}_{a,j} + C^b{}_a|^\alpha \Gamma_{\alpha 0 j} = 0.$

Similarly, from (5.1), (5.2) and (4.4) we have $C^b{}_a|^i = 0$, that is $C^b{}_a{}^{|i} = 0$, considering these relations and (5.9), we have

Theorem [5.2]. *If the space of absolute parallelism admits an infinitesimal transformations* (3.1) *satisfying the equations* (5.1), *the Lie derivatives of the n independent vectors of an absolutely parallel ennuple are expressed by linear combinations of them with constant coefficients.*

If the transformation (3.1) satisfies the conditions $X\Gamma_j{}^i{}_k = 0$, then we may say, following K.Yano[1] and M.S.Knebelman[2], the transformation is an *affine motion.*

Let $\xi^i{}_{(\alpha)}$ and $\xi^i{}_{(\beta)}$ being the components of two linearly independent vectors giving the infinitesimal affine motions in our space, then the components of their alternations are $\xi^m{}_{(\beta)}\xi^i{}_{(\alpha);m} - \xi^m{}_{(\alpha)}\xi^i{}_{(\beta);m} = X_{(\beta)}\xi^i{}_{(\alpha)}$, in our notation $\xi^i = X_\beta \xi^i{}_{(\alpha)}$.

We wish to proce that, ξ^i are the components of an vector giving an affine motion.

From the conditions $X_{(\alpha)}\Gamma_j{}^i{}_k = 0$ and $X_{(\beta)}\Gamma_j{}^i{}_k = 0$, considering theorems [4.1] and [4.5], for ξ^i the equations (3.9) may be written in the form

$$X\Gamma_j{}^i{}_k = (X_{(\beta)}\Gamma_k{}^i{}_j|^\alpha l_\rho + \Gamma_k{}^i{}_j|^\alpha X_{(\beta)} l_\rho)\xi^\rho{}_{(\alpha);\alpha}.$$

From (4.5) and (3.7), the right side of above equation vanishes, then we have

Theorem [5.3]. *If $D_\alpha f$ for $\alpha = 1,2,\cdots,p$ are generators of p one-parameter groups of the affine motions, so also are each of the commutators $(D_\alpha D_\beta)f$, for $\alpha,\beta = 1,2,\cdots,p$.*

1) See [10], §3
2) See [8], §2

From this result and the fundamental theorems of continuous groups we have

Theorem 〔5.4〕. *If* $D_{(\alpha)}f$ *for* $\alpha=1,\cdots,p$ *are generators of the complete set of one-parameter groups of affine motions, they are the generators of a group* G_p *of affine motions.*

(March, 1952)

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