Connection Parameters in a Special Kawaguchi Space

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Connection Parameters in a Special Kawaguchi Space

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叶 長太郎：特殊河口空間における接続経数について

Introduction

Geometry in a special Kawaguchi space with the arc length,

$$S=\int\{A_i(x,x'')x'''^i+B(x,x',x'')\}^{1/p}dt,$$

was studied by T. Ohkubo [1].

In the present paper we shall introduce the other covariant derivative which is invariant for a change of parameter. In this space we have the Synge vectors

$$\overset{2}{E}_i=3A_{i(1)j}\,x'^j+3\mathit{\Lambda}_i(x,x',x''),$$
$$\overset{1}{E}_i=G_{ij}\,x'''^j+\Gamma_i(x,x',x''),$$

where

$$G_{ij}=3A_{i(1)j}+A_{j(1)i}-2B_{(2)i(2)j},$$
$$\mathit{\Lambda}_i=A_{i(0)j}\,x'^j-\frac{1}{3}B_{(2)i},$$
$$\Gamma_i=B_{(1)i}-2B_{(2)i(1)j}\,x''^j-2B_{(2)i(0)j}\,x'^j+3(A_{i(1)j(1)k}\,x''^j\,x''^k$$
$$+2A_{i(1)j(0)k}\,x''^j\,x'^k+A_{i(0)j(0)k}x'^kx'^j+A_{i(0)j}x''^j).$$

Then we have

$$\frac{1}{3}A^{ik}\,\overset{2}{E}_i=x''^k+\mathit{\Lambda}^k,\qquad \mathit{\Lambda}^k=A^{ik}\mathit{\Lambda}_i,$$
$$G^{ik}\,\overset{1}{E}_i=x'''^k+\Gamma^k,\qquad \Gamma^k=G^{ik}\,\Gamma_i,$$
$$A^{ik}A_{i(1)j}=\delta^k_j,\qquad G^{ik}\,G_{ik}=\delta^k_j.$$

By use of $\mathit{\Lambda}^i$ and Γ^i, we shall introduce the linear connection which is invariant for a change of parameter t.

§1. Synge vector

In the space we have *Zermelo's conditions*

$$\sum_{\beta=\rho}^{3}\binom{\beta}{\rho}x^{(\beta-\rho+1)i}\,F_{(\beta)i}=p\,\delta^1_\rho\,F,\ (\rho=1,2,3),$$

that is,

(1.1) $\quad A_i\,x'^i=0,\qquad A_{i(1)j}\,x'^j=(p-3)A_i,$

$\quad B_{(2)i}x'^i+3A_i\,x''^i=0,\qquad B_{(1)i}\,x'^i+2B_{(2)i}\,x''^i=pB.$

From (1.1) we have

(1.2) $\quad A_{i(1)j}\,x'^i=-A^j,\qquad A_{i(0)j}\,x'^i=0,\qquad B_{(2)i(2)j}x'^i=-3A_j.$

Furthermore, using (1.1) and (1.2), after some calculation we have

(1.3) $\Gamma_l x'^l = pB,$

(1.4) $\Gamma_{l(2)j} x'^j = -3G_{lk} x''^k + (2p-3)\overset{2}{E}_l,$

(1.5) $\Gamma_{l(2)j} x'^j = -3x'''^l + (2p-3)\overset{2}{E}{}^l, \qquad \overset{2}{E}{}^l = G^{jl}\overset{2}{E}_j,$

(1.6) $\overset{2}{E}{}^l_{(2)j} x'^j = \dfrac{3(p-2)}{3p-4} x'^l,$

(1.7) $A^l_{(2)j} x'^j = \dfrac{1}{p-3} x'^l.$

Now, for a change of parameter t, we have

(1.8) $A_l(\bar{t}) = \gamma^{p-3} A_l, \qquad A_{l(1)j}(\bar{t}) = \gamma^{p-4} A_{l(1)j}, \qquad A_{l(1)j(1)k}(\bar{t}) = \gamma^{p-5} A_{l(1)j(1)k},$

$B_{(2)l(2)j}(\bar{t}) = \gamma^{p-4} B_{(2)l(2)j}, \qquad G_{lj}(\bar{t}) = \gamma^{p-4} G_{lj}, \qquad G^{lj}(\bar{t}) = \gamma^{-p+4} G^{lj}.$

Next, using (1.8) we obtain

(1.9) $\overset{2}{E}_l = \gamma^{p-2}\overset{2}{E}_l + \gamma^{p-4}\dfrac{d\gamma}{dt} 3(p-2) A_l,$

(1.10) $\overset{1}{E}_l = \gamma^{p-1}\overset{1}{E}_l + S\gamma^{p-3}\dfrac{d\gamma}{dt}\overset{2}{E}_l - \left\{\gamma^{p-4}\dfrac{d^2\gamma}{dt^2} Q - \gamma^{p-5}\left(\dfrac{d\gamma}{dt}\right)^2 R\right\}\overset{3}{E}_l$

where

$$S = 2p-3, \qquad Q = -3p+4, \qquad R = 3(p^2-5p+5).$$

Then, using (1.8) and (1.9) we have

(1.11) $K(\bar{t}) = \gamma K + \gamma^{-1}\dfrac{d\gamma}{dt}$

where

$$K = G^{lj} G^{lk} A_{l(1)j(1)k}\overset{2}{E}_l \Big/ \frac{3(p-2)(p-4)}{3p-4} G^{lj} A_{l(1)j},$$

or

$$K = G^{lj} A^{lk} A_{l(1)j(1)k}\overset{2}{E}_l \Big/ \frac{3(p-2)(p-4)}{p-3} G^{lj} A_{l(1)j}.$$

At last, from (1.10), we obtain

(1.12) $\Gamma^j(\bar{t}) = \gamma^3\Gamma^j + \gamma^3 x^{(3)j} - x^{\bar{(3)}j} + \gamma\dfrac{d\gamma}{dt} S G^{lj}\overset{2}{E}_l - \left\{\gamma^{p-4}\dfrac{d^2\gamma}{dt^2} Q - \gamma^{p-5}\left(\dfrac{d\gamma}{dt}\right)^2 R\right\} G^{lj}\overset{3}{E}_l,$

(1.13) $\overset{2}{E}{}^l = \gamma^2\overset{2}{E}{}^l + \dfrac{3(p-2)}{3p-4} x'^l \dfrac{d\gamma}{dt}.$

Differentiating (1.12) and (1.13) with respect to $x^{\bar{(2)}k}$, we have

(1.14) $\Gamma^l_{(2)k}(\bar{t}) = \gamma\,\Gamma^l_{(2)k} + \Gamma^l_{(2)j(2)k} x'^k \gamma^{-1}\dfrac{d\gamma}{dt},$

(1.15) $\overset{2}{E}{}^l_{(2)k}(\bar{t}) = \overset{2}{E}{}^l_{(2)k}.$

§ 2. Connection parameter (I)

Since $1/3\, A^{lk}\overset{2}{E}_l$ is the vector, we have

$$\bar{x}'''^l + \bar{A}^l(x, x'', x''') = \frac{\partial \bar{x}^l}{\partial x^k}(x''^k + A^k), \qquad \bar{x}''''^l + \Gamma^l(x, x', x'') = \frac{\partial \bar{x}^l}{\partial x^k}(x'''^k + \Gamma^k)$$

for the coordinate transformation.

Hence we have

(2.1) $$\bar{A}^i=\frac{\partial \bar{x}^i}{\partial x^k}A^k+\frac{\partial \bar{x}^i}{\partial x^k}x''^k-x''^i,$$

(2.2) $$\bar{\Gamma}^i=\frac{\partial \bar{x}^i}{\partial x^k}\Gamma^k+\frac{\partial \bar{x}^i}{\partial x^k}x'''^k-x'''^i.$$

Differentiating (2.1) and (2.2) with respect to x'^j and x''^j respectively, we have

(2.3) $$\bar{\Gamma}^i_{(2)j}=\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}\Gamma^k_{(2)l}+3\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial^2 x^k}{\partial \bar{x}^j\partial \bar{x}^n}\bar{x}'^n,$$

(2.4) $$\bar{A}^i_{(1)j}=\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}A^k_{(1)l}+2\frac{\partial \bar{x}^i}{\partial x^k}\left(\frac{\partial x^k}{\partial \bar{x}^j}\right)^{(1)}+\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x''^l}{\partial \bar{x}'^j}A^k_{(2)l}.$$

Substituting (2.3) to (2.4) we have

$$\left(\bar{A}^i_{(1)j}-\frac{2}{3}\bar{\Gamma}^i_{(2)j}\right)\frac{\partial x^k}{\partial \bar{x}^i}=\frac{\partial x^l}{\partial \bar{x}^j}M_l{}^k+\frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial x''^p}{\partial \bar{x}'^j}A^l_{(2)p},$$

where

$$M^i_j=M^i_{(1)j}-\frac{2}{3}\Gamma^i_{(2)j}.$$

From $A^l_{(2)p}\neq 0$, we have by (1.7)

(2.5) $$\Omega^m_{\cdot k}\,\Omega_l^{\cdot k}=\delta^m_l,\qquad \Omega^k_{\cdot l}\,\Omega_k^{\cdot j}=\delta^j_l,\qquad \Omega^m_{\cdot k}\,A_m=-A_k,\qquad \Omega^m_{\cdot k}=A^m_{(2)k},$$

(2.6) $$\frac{\partial x^m}{\partial \bar{x}^l}\bar{M}^l_j\,\Omega_m^{\cdot i}=\frac{\partial x^k}{\partial \bar{x}^j}M^m_k\,\Omega_m^{\cdot i}+2\frac{\partial^2 x^i}{\partial \bar{x}^j\partial \bar{x}^n}\bar{x}'^n.$$

On the other hand, differentiating (2.3) with respect to x'^k we have

$$\begin{aligned}\bar{\Gamma}^i_{(2)j(1)k}&=\frac{\partial \bar{x}^i}{\partial x^m}\frac{\partial x^l}{\partial \bar{x}^j}\Gamma^m_{(2)l(1)q}\frac{\partial x^q}{\partial \bar{x}^k}+3\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial^2 x^p}{\partial \bar{x}^j\partial \bar{x}^k}+\frac{\partial \bar{x}^i}{\partial x^m}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x''^q}{\partial \bar{x}'^k}\Gamma^m_{(2)l(2)q}\\&=\frac{\partial \bar{x}^i}{\partial x^m}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m_{(2)l(1)q}+3\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial^2 x^p}{\partial \bar{x}^j\partial \bar{x}^k}\\&\quad+\frac{\partial \bar{x}^i}{\partial x^m}\frac{\partial x^l}{\partial \bar{x}^j}\Gamma^m_{(2)l(2)q}\left(\bar{M}^t_k\,\Omega_r^{\cdot q}\frac{\partial x^r}{\partial \bar{x}^t}-M^t_r\,\Omega_t^{\cdot i}\frac{\partial x^r}{\partial \bar{x}^k}\right).\end{aligned}$$

Seeing that $\Gamma^i_{(2)j(2)k}$ is a tensor, we have

(2.7) $$\bar{\Pi}^i_{jk}=\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^t}\Pi^t_{pq}-\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial^2 \bar{x}^i}{\partial x^p\partial x^q},$$

where

$$3\,\Pi^i_{jk}(x,x',x'')=\Gamma^i_{(2)j(1)k}-\Omega_p^{\cdot q}\,M^p_k\,\Gamma^i_{(2)j(2)q}.$$

Thus, using (2.7) we can define *the covariant differential*

$$\delta v^i=dv^i+\Pi^i_{jk}\,v^j\,dx^k.$$

The base connection is given by

$$\delta x'^i=dx'^i+\Pi^i_{jk}\,x'^j\,dx^k,\qquad \overset{(2)}{x}{}^i=x''^i+\Pi^i_{jk}\,x'^j\,x'^k,$$

$$\delta\overset{(2)}{x}{}^i=d\overset{(2)}{x}{}^i+\Pi^i_{jk}\,\overset{(2)}{x}{}^j\,dx^k,\qquad \overset{(3)}{x}{}^i=\overset{(2)}{x}{}'+\Pi^i_{jk}\,\overset{(2)}{x}{}^j\,x'^k.$$

§ 3. Connection parameter (II)

Substituting (2.3) to the third term in the right member of (2.4), we have

(3.1) $$\bar{A}^i_j=\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^q}A^q_p-2\frac{\partial x^t}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^t\partial x^p}x'^p,$$

or

$$(3.1)' \qquad \frac{\partial \bar{x}^i}{\partial x^k} A^i_j = \frac{\partial \bar{x}^i}{\partial x^q} A^q_k - 2 \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^p} x'^p,$$

where

$$(3.2) \qquad A^i_j = A^i_{(1)j} - \frac{2}{3} \Gamma^p_{(2)j} A^i_{(2)p}.$$

After short calculation we obtain

$$(3.3) \qquad A^i_j x'^j = 2A^i - \frac{2(2p-3)}{3} \overset{2}{E}{}^p A^i_{(2)p} = 2A^i - \frac{2(2p-3)}{3} A^i_{(2)j} x''^j.$$

Differentiating (3.1) with respect to x'^k and using (3.1), we have

$$(3.4) \qquad \bar{A}^i_{jk} = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^q} \frac{\partial x^t}{\partial \bar{x}^k} A^q_{pt} - 2 \frac{\partial x^t}{\partial \bar{x}^k} \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^t},$$

or

$$(3.4)' \qquad \bar{A}^i_{jk} \frac{\partial \bar{x}^j}{\partial x^t} \frac{\partial \bar{x}^k}{\partial x^s} = \frac{\partial \bar{x}^i}{\partial x^q} A^q_{ts} - 2 \frac{\partial^2 \bar{x}^i}{\partial x^t \partial x^s},$$

where

$$(3.5) \qquad A^i_{jk} = A^i_{j(1)k} - A^i_{j(2)s} A^s_k.$$

Thus, using (3.4) we can define the covariant differential, but we will modify.

Now, if we put

$$*A^i_j = A^i_j + \frac{2(2p-3)}{q(p-4)} A^{pt}_{(1)j} B_{(2)p(2)q} \overset{2}{E}{}^q,$$

then the transformation of $*A^i_j$, for the coordinate transformation, is (3.1), because the second term in the above equation is a tensor. Using (3.3) we have $*A^i_j x'^j = 2A^i$. Hence, by means of the process which obtained (3.4), $*A^i_{jk} = *A^i_{j(1)k} - *A^i_{j(2)s} *A^s_k$ have the same transformation law as the case of A^i_{jk}.

Thus we can define *the covariant differential*

$$\delta v^i = dv^i + *A^i_{jk} v^j dx^k.$$

§ 4. Connection parameter (III)

In § 2 and § 3, by use of A^k and Γ^k we have introduced the connection parameter Π^i_{jk}, $*A^i_{jk}$. In this paragraph, from only Γ^i we shall introduce a connection parameter which is not intrinsic. Differentiating (2.3) with respect to x'^k we have

$$\bar{\Gamma}^i_{(2)j(1)k} = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \left(\frac{\partial \bar{x}^i}{\partial x^t} \Gamma^t_{(2)p(1)q} - \frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^q} \right) + \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^t} \frac{\partial x''^q}{\partial \bar{x}'^k} \Gamma^t_{(2)p(2)q}.$$

Since $\Gamma^t_{(2)p(2)q}$ is a tensor, using (2.3) we have

$$(4.1) \qquad \bar{\Gamma}^i_{jk} = \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^t} \Gamma^t_{pq} - \frac{\partial x^p}{\partial \bar{x}^q} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^q},$$

where

$$(4.2) \qquad 3\Gamma^i_{jk} = \Gamma^i_{(2)j(1)k} - \frac{2}{3} \Gamma^i_{(2)j(2)t} \Gamma^t_{(2)k}.$$

Thus, from (4.1), we define The *covariant differentiation*

(4.3) $\delta v^i = dv^i + \Gamma^i_{jk} v^j dx^k$

and *the base connection*

(4.4) $$\delta x'^i = dx'^i + \Gamma^i_{jk} x'^j dx^k, \qquad \overset{(2)}{x}{}^i = x''^i + \Gamma^i_{jk} x'^j x'^k,$$
$$\delta \overset{(2)}{x}{}^i = d\overset{(2)}{x}{}^i + \Gamma^i_{jk} \overset{(2)}{x}{}^j dx^k, \qquad \overset{(3)}{x}{}^i = \overset{(2)}{x}{}'^i + \Gamma^i_{jk} \overset{(2)}{x}{}^j x'^k.$$

§ 5. Intrinsic connection parameter

In this paragraph we shall introduce *the intrinsic connection parameter.*

Theorem 5.1. *If we define*

$$^*\underset{1}{\prod}{}^i_{jk} = \Gamma^i_{jk} - \Omega^{\cdot u}_v M^v_k \Gamma^i_{j(2)u}, \qquad \Gamma^i_{jk} = \Gamma^i_{j(1)k} + K\Gamma^i_{j(2)k},$$

$$\Gamma^i_j = \frac{1}{3}(\Gamma^i_{(2)j} - K\Gamma^i_{(2)k(2)j} x'^k), \qquad N^i_j = \underset{1}{A}{}^i_j - 2\Gamma^i_j,$$

$$\underset{1}{A}{}^i_j = A^i_{(1)j} - \frac{1}{p-3}(\delta^i_j - A^i_{(1)j})K,$$

then $^*\underset{1}{\prod}{}^i_{jk}$ *is the intrinsic connection parameter.*

Proof: From (1.11) and (1.14), we have

(5.1) $\Gamma^i_j(\bar{t}) = \gamma\, \Gamma^i_j$

for a change of parameter and

(5.2) $$\bar{\Gamma}^i_j = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^q}\Gamma^q_p - \frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^q} x'^p$$

for the coordinate transformation. Differentiating (5.1) with respect to x'^k and using (1.11), we have

(5.3) $$\Gamma^i_{j(1)k}(\bar{t}) + K(\bar{t})\Gamma^i_{j(2)k}(\bar{t}) = \Gamma^i_{j(1)k} + K\Gamma^i_{j(2)k}$$

i.e., $\Gamma^i_{jk}(\bar{t}) = \Gamma^i_{jk}$.

Further, differentiating (5.2) with respect to x'^k we have

$$\bar{\Gamma}^i_{j(1)k} = \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^t}\Gamma^t_{p(1)q} - \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^t \partial x^q} + 2\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^t}\Gamma^t_{p(2)q}\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^t}\bar{x}'^t.$$

Since $K\Gamma^i_{j(2)k}$ is a tensor, we obtain

(5.4) $$\bar{\Gamma}^i_{jk} = \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^t}\Gamma^t_{pq} - \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^t \partial x^q} + 2\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^t}\Gamma^t_{p(2)q}\frac{\partial^2 x^q}{\partial \bar{x}^k \partial \bar{x}^n}\bar{x}'^n.$$

On the other hand, we have

$$\bar{\underset{1}{A}}{}^i_j = \frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^p}\underset{1}{A}{}^p_q - 2\frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^t}x'^t + 2\frac{\partial^2 x^t}{\partial \bar{x}^j \partial \bar{x}^n}\frac{\partial \bar{x}^i}{\partial x^q}\bar{x}'^n A^q_{(2)t}.$$

Substituting (5.2) to the above equation we have

(5.5) $$\bar{N}^i_j = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^q}N^q_p + 2\frac{\partial^2 x^t}{\partial \bar{x}^j \partial \bar{x}^n}\bar{x}'^n\frac{\partial \bar{x}^i}{\partial x^q}A^q_{(2)t}.$$

Multiplying (5.5) with $\Omega^{\cdot t}_q$ and summing with q we have

(5.6) $$\bar{N}^p_j \Omega^{\cdot t}_q \frac{\partial x^q}{\partial \bar{x}^p} = \frac{\partial x^p}{\partial \bar{x}^j}N^q_p \Omega^{\cdot t}_q + 2\frac{\partial^2 x^t}{\partial \bar{x}^j \partial \bar{x}^t}\bar{x}'^t.$$

Thus, substituting (5.6) to (5.4) we obtain

(5.7) $$\overset{*}{\underset{1}{\overline{\prod}}}{}^{i}_{jk}=\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^t}\,{}^{*}\underset{1}{\prod}{}^{t}_{pq}-\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial^2 \bar{x}^i}{\partial x^p\partial x^q}.$$

Further, from

$$\Omega_v^{;u}(\bar{t})=\Omega_v^{;u},\qquad N_j^i(\bar{t})=\gamma\, N_j^i,\qquad \Gamma^i_{j(2)k}(\bar{t})=\gamma^{-1}\Gamma^i_{j(2)k}$$

we have

$$^{*}\underset{1}{\prod}{}^{i}_{jk}(\bar{t})={}^{*}\underset{1}{\prod}{}^{i}_{jk}(t)$$ Q.E.D.

By means of Theorem 5.1 we can define the intrinsic covariant differential

(5.8) $$\delta v^i=dv^i+{}^{*}\underset{1}{\prod}{}^{i}_{jk}\,v^j\,dx^k.$$

Theorem 5.2 : If we define

$$2{}^{*}\underset{2}{\prod}{}^{i}_{jk}={}^{*}\underset{1}{A}{}^{i}_{jk}-{}^{*}\underset{1}{A}{}^{i}_{j(2)s}\,{}^{*}\underset{1}{A}{}^{s}_{k},\qquad {}^{*}\underset{1}{A}{}^{i}_{jk}={}^{*}\underset{1}{A}{}^{i}_{j(1)k}+K\,{}^{*}\underset{1}{A}{}^{i}_{j(2)k},\qquad {}^{*}\underset{1}{A}{}^{i}_{j}=\underset{1}{A}{}^{i}_{j}-2\Gamma^p_j\,A^i_{(2)p}$$

then $^{*}\underset{2}{\prod}{}^{i}_{jk}$ *is an intrinsic connection parameter.*

Proof : Both $\underset{1}{A}{}^i_j$ and $A^i_{(1)j}$ are given the same transformation, and hence we have

(5.9) $$\underset{1}{\bar{A}}{}^i_j=\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^p}\underset{1}{A}{}^p_q-2\frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^k\partial x^t}x'^t+2\frac{\partial^2 x^t}{\partial \bar{x}^j\partial \bar{x}^s}\bar{x}'^s\frac{\partial \bar{x}^i}{x x^q}A^q_{(2)t}$$

for the coordinate transformation and

(5.10) $$\underset{1}{\bar{A}}{}^i_j(\bar{t})=\gamma \underset{1}{A}{}^i_j(t)$$

for a change of parameter. From (5.2) and (5.9), we have

(5.11) $$^{*}\underset{1}{\bar{A}}{}^i_j\frac{\partial \bar{x}^j}{\partial x^k}=\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^q}\,{}^{*}\underset{1}{A}{}^q_p-2\frac{\partial^2 \bar{x}^i}{\partial x^p\partial x^t}x'^t$$

From (5.10) we obtain

(5.12) $$^{*}\underset{1}{A}{}^i_{j(2)k}=\gamma^{-1}\,{}^{*}\underset{1}{A}{}^i_{j(2)k},\qquad {}^{*}\underset{1}{A}{}^i_{j(1)k}+K\,{}^{*}\underset{1}{A}{}^i_{j(2)k}={}^{*}\underset{1}{A}{}^i_{j(1)k}+K\,{}^{*}\underset{1}{A}{}^i_{j(2)k}$$

and, since $K\,{}^{*}\underset{1}{A}{}^i_{j(2)k}$ is a tensor, we have

$$^{*}\underset{1}{\bar{A}}{}^i_{jk}=\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^q}\frac{\partial x^t}{\partial \bar{x}^k}\,{}^{*}\underset{1}{A}{}^q_{pt}-2\frac{\partial x^m}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^t\partial x^m}+2\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^i}{\partial \bar{x}^q}\frac{\partial^2 \bar{x}^t}{\partial \bar{x}^s\partial \bar{x}^k}\bar{x}'^s\,{}^{*}A^q_{p(2)t}.$$

At last, substituting (5.11) to the above equations, we have

$$^{*}\underset{2}{\overline{\prod}}{}^i_{jk}=\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^q}\,{}^{*}\underset{2}{\prod}{}^q_{pt}-2\frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^p\partial x^q}.$$

Further, from (5.12) and (5.10), we have

$$^{*}\underset{2}{\overline{\prod}}{}^i_{jk}(\bar{t})={}^{*}\underset{2}{\prod}{}^i_{jk}(t)$$ Q.E.D.

Hence we obtain *the covariant differential*

$$\delta v^i=dv^i+{}^{*}\underset{2}{\prod}{}^i_{jk}\,v^j\,dx^k.$$

Theorem 5.3 If we define

$$^{*}\underset{3}{\prod}{}^i_{jk}=\Gamma^i_{j(1)k}+K\,\Gamma^i_{j(2)k}-\frac{2}{3}\Gamma^i_{j(2)t}\,\Gamma^t_k$$

then $^{*}\underset{3}{\prod}{}^i_{jk}$ *is an intrinsic connection parameter.*

Proof : Differentiating (5.2) with respect to x'^k and using (5.2), we have

(5.13) $$\bar{\Gamma}^i_{j(1)k}-2\bar{\Gamma}^p_k\,\bar{\Gamma}^i_{j(2)p}=\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^t}\left(\Gamma^t_{p(1)q}-2\Gamma^s_q\,\Gamma^t_{p(2)s}\right)-\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^j}\frac{\partial^2 \bar{x}^i}{\partial x^t\partial x^q}.$$

Since $\Gamma^p_k \Gamma^i_{j(2)p}$ is an intrinsic tensor, making use of (1.11) we have

$$(\Gamma^i_{j(1)k} - 2\Gamma^p_k \Gamma^i_{j(2)p})(\bar{t}) = \Gamma^i_{j(1)k} - 2\Gamma^p_k \Gamma^i_{j(2)p} - \gamma^{-2}\frac{d\gamma}{dt}\Gamma^i_{j(2)k}$$
$$= \Gamma^i_{j(1)k} - 2\Gamma^p_k \Gamma^i_{j(2)p} - (K(\bar{t}) - \gamma K)\gamma^{-1}\Gamma^i_{j(2)k},$$

and consequently

(5.14) $\quad {}^*\prod^i_{jk}(\bar{t}) = {}^*\underset{3}{\prod}{}^i_{jk}(t)$

Thus, since $K\Gamma^i_{j(2)k}$ is a tensor, by use of (5.13) and (5.14), we have

$${}^*\underset{3}{\bar{\prod}}{}^i_{jk} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^t}{}^*\underset{3}{\prod}{}^t_{pq} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^q} \qquad \text{Q.E.D.}$$

§ 6. Base connection

We shall introduce *the base connection* in the special Kawaguchi space.

The expressions

(6.1) $\quad \underset{\alpha}{\overset{(1)}{\delta x^i}} = d(x'^i/F)^* + \underset{\alpha}{\prod}{}^i_{jk}(x'^j/F)dx^k, \qquad \underset{\alpha}{\overset{(2)}{\delta x^i}} = \overset{(2)}{dx^i} + {}^*\underset{\alpha}{\prod}{}^i_{jk}\overset{(2)}{x^j}\, dx^k, \qquad \alpha = 1, 2, 3,$

are the components of a contravariant vector. The expressions (6.1) give the base connection in K^2_n, a special Kawaguchi space of order 2 and dimension n.

Along the base curve the equations (6.1) become

(6.2) $\quad \underset{\alpha}{\overset{(2)}{\delta x^i}} = \frac{d}{dt}(x'^i/F) + {}^*\underset{\alpha}{\prod}{}^i_{jk}x'^j x'^k F^{-1}, \qquad \overset{(3)}{x^i} = \frac{d}{dt}\overset{(2)}{x^i} + {}^*\underset{\alpha}{\prod}{}^i_{jk}\overset{(2)}{x^j}x'^k.$

Thus, making use of (6.1) and (5.2), we have the covariant derivative.

Lately the author have introduced the intrinsic covariant differential in the special Kawaguchi space with arc length

$$s = \int \{A_i(x, x', ..., x^{(m-2)i})x^{(m)i} + B(x, x', ... x^{(m-1)i})\}^{1/p} dt \quad [4].$$

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