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On the Conformal Differential in a Higher Order Space

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叶 長太郎：高次空間における共形微分について

Introduction

In the previous paper [4], by use of a non-intrinsic conformal derivative, we have given an intrinsic conformal connection in the space with the metric $s=\int(A_i(x,x')x''^i+B(x,x'))^{1/p}dt$.

In this paper, by use of a intrinsic conformal derivative, we shall give the same conformal connection.

In the first place we shall give an intrinsic conformal derivative and the effect of this operation for relative conformal tensors. In the second place, making use of this intrinsic conformal derivative we shall have the intrinsic conformal differential. At last we shall give the base connection and the conformal covariant derivative which is intrinsic for a change of parameter.

§ 1. Intrinsic conformal derivative

In the previous paper [3], we have defined a conformal derivative for any relative conformal tensor of weight k, as follows:

$$(1.1)\qquad \varphi_j(V)=V_{(1)j}-2V_{(2)q}H^q_{(1)j}-kF^{-1}A_{p(j)}x^{|2|p}V.$$

Let V go into $\gamma^h V$ by transformation of parameter t. Then, for a change of parameter, we have

$$(1.2)\qquad \varphi_j(V)(\bar{t})=\gamma^{h-1}\varphi_j(V)-\gamma^{h-3}\frac{d\gamma}{d\bar{t}}(V_{(2)j}-kA_jVF^{-1}),$$

$$\{V_{(2)j}-kA_jVF^{-1}\}(\bar{t})=\gamma^{h-2}(V_{(2)j}-kA_jVF^{-1}).$$

On the other hand we have

$$(1.3)\qquad \mu(\bar{t})=\gamma\mu-2\gamma^{-1}\frac{d\gamma}{d\bar{t}}.$$

where

$$\gamma=\frac{dt}{d\bar{t}},\qquad \mu(x,\ x',\ x'')=(M_{(j)(k)}x^{|2|j}x^{|2|k})(M_{(l)}x^{|2|l})^{-1},\quad [3].$$

From (1.2) and (1.3) we obtain *the intrinsic conformal derivative :*

$$(1.4)\qquad \phi_j(V)=\varphi_j(V)-\frac{1}{2}\mu(V_{(2)j}-kF^{-1}A_jV).$$

Thus we have

$$\phi_j(\widetilde{V}) = e^{kp\sigma}\phi_j(V), \quad \phi_j(V)(\bar{t}) = \gamma^{h-1}\phi_j(V).$$

When $\overset{1}{V}$ and $\overset{2}{V}$ are relative conformal tensors of the same type, we obtain

$$\phi_j(a\overset{1}{V} + b\overset{2}{V}) = a\phi_j(\overset{1}{V}) + b\phi_j(\overset{2}{V}), \tag{1.5}$$

where a and b are constant. Further, when both V and W are any relative conformal tensor and P-tensor [5], we obtain

$$\phi_j(V \cdot W) = \phi_j(V) \cdot W + V \cdot \phi_j(W). \tag{1.6}$$

After short calculation we have

$$\{\phi_j(V)\}_{(2)k}(\bar{t}) = \phi_j(V_{(2)k}) - \frac{1}{2}\mu_{(2)k}V_{(2)j} - \frac{1}{2}kF^{-1}V(\mu A_k F^{-1} + \mu_{(2)k})A_j. \tag{1.7}$$

Specially it can be verified that

$$\begin{aligned} &\phi_j(x'^k) = \delta^k_j, \quad \phi_j(F^\alpha) = 0, \quad \phi_j(\delta^i_k) = 0, \\ &\phi_j(A_i) = \varphi_j(A_i) + \frac{1}{2}\mu F^{-1}A_i A_j, \quad \phi_j(A_i)x'^j = 0, \quad \phi_j(A_i)x'^i = -A_j. \end{aligned} \tag{1.8}$$

Applying (1.1) to the equation $A_i x^{|2|i} = A_i X^i = F$, we have

$$\phi_j(A_i)X^i = -A_i\phi_j(X^i) = 0, \tag{1.9}$$

where

$$X^i = x^{|2|i} + \frac{1}{2}\mu x'^i, \quad \phi_j(X^i) = \frac{1}{2}(\varphi_j(\mu) - \mu_{(2)j})x'^i.$$

On the other hand we have

$$\phi_j(A_i)X^j = \varphi_j(A^i)x^{|2|j} + \frac{1}{2}\mu A_i = A^*_i. \tag{1.10}$$

Next, differentiating (1.3) with respect to $x^{(\bar{2})p}$ we have

$$\mu_{(2)p}(\bar{t}) = \gamma^{-1}\mu_{(2)p}.$$

Further, after short calculation, we have

$$\mu_{(2)p}x'^p = -2, \quad \mu_{(2)p}X^p = 0. \tag{1.11}$$

Thus, by virtue of (1.4), (1.6), (1.7) and (1.11), we have

$$T^*_{i(2)j} = -2\phi_j(A_i) + 2\phi_k(A_j)X^k A_i F^{-1}, \tag{1.12}$$

where

$$T^*_i = -2\varphi_k(A_i)x^{|2|k} = -2A^*_i + \mu A_i.$$

Then, from (1.8), (1.9) and (1.11), we have the following relations;

$$T^*_{i(2)j}x'^j = -2A_i, \quad T^*_{i(2)j}x'^i = 2A_j. \tag{1.13}$$

Further, applying (1.1) to (1.8) and using (1.8), we have

$$\phi_k\phi_j(A_i)x'^j = -\phi_k(A_i), \quad \phi_k\phi_j(A_i)x'^k = -\phi_j(A_i), \quad \phi_k\phi_j(A_i)x'^i = -\phi_j(A_i) - \phi_k(A_j). \tag{1.14}$$

§ 2. Intrinsic conformal differential

In the first place we shall give

$$S^{*ik}\phi_n(A_i) = -\frac{1}{2}(\delta^k_n + \phi_p(A_n)X^p x'^k F^{-1}). \tag{2.1}$$

Applying (1.1) to $S^{*ik}T^*_{i(2)j} = \delta^k_j$ and contracting the resultant equations with x'^j, we have, by virtue of (1.6), (1.13), (1.8) and (1.10),

$$A_l\phi_n(S^{*lk})=\frac{1}{2}\phi_p(A_n)X^px'^kF^{-1}. \tag{2.2}$$

On the other hand, applying (1.1) to $A_lS^{*lk}=-\frac{1}{2}x'^k$, we obtain

$$S^{*lk}\phi_n(A_l)=-\frac{1}{2}\delta^k_n-A_l\phi_n(S^{*lk}).$$

Substituting (2.2) for the above equations, we have (2.1).
Thus, applying (1.1) to (2.1), we have

$$\phi_n(S^{*mj})\phi_k(A_m)=-S^{*mj}\phi_n\phi_k(A_m)-\frac{1}{2}\phi_n\phi_p(A_k)X^px'^jF^{=1}-\frac{1}{2}\phi_p(A_k)X^p\delta^j_nF^{-1}. \tag{2.3}$$

If we put

$$D^{*j}_{kn}=\phi_n(S^{*mj})\phi_k(A_m),$$

it follows that

$$D^{*j}_{kn}x'^k=0,\quad D^{*j}_{kn}x'^n=S^{*mj}\phi_k(A_m), \tag{2.4}$$

and

$$\widetilde{D}^{*j}_{kn}=D^{*j}_{kn},\quad D^{*j}_{kn}(\bar{t})=D^{*j}_{kn}. \tag{2.5}$$

Further, on account of (2.4) if we put

$$D'^j_{kn}=D^{*j}_{kn}+S^{*mj}\phi_k(A_m)\phi_p(A_n)X^pF^{-1},$$

then we have

$$D'^j_{kn}x'^k=0,\quad D'^j_{kn}(\bar{t})=D'^j_{kn},$$

and

$$\widetilde{D}'^j_{kn}=D'^j_{kn},\quad D'^j_{kn}x'^n=0.$$

Thus, with the help of the conformal tensor D'^j_{kn}, *an absolute differential* can be defined by

$$\overset{\mathrm{I}}{\delta}v^i=dv^i+H^i_{(k)(l)}v^kdx^l+D'^i_{kl}v^k\delta^*x'^l,$$

which is invariant for a change of parameter t, and this defferential coincides with the intrinsic conformal differential $D''v^i$ in the previous paper [4].

§ 4. Base connection

Let v^i be a relative conformal vector, which is $\widetilde{v}^i=e^{kp\sigma}v^i$ for the conformal transformation, and let v^i go into γ^hv^i by transformation of parameter t. In § 2 we have obtained the intrinsic conformal differential, as follows;

$$\overset{\mathrm{I}}{\delta}v^i=dv^i+\Lambda^i_jv^j+\Lambda v^i, \tag{3.1}$$

where

$$\Lambda^i_j=H^i_{(j)(k)}dx^k+D''^i_{jk}\delta^*x'^k,\quad \Lambda(h,k)=\{h-(p-2)k\}\nu-kQ_jX^jF^{-1}, \tag{3.2}$$

$$\nu=\frac{1}{2}\mu_{(2)k}\delta^*x'^k.$$

From (3.1) the expressions

$$\overset{\mathrm{I}(1)}{\delta x^i}=dx'^i+\Lambda^i_jx'^j+\Lambda(1,0)x'^i,\quad \overset{\mathrm{I}(2)}{\delta x^i}=dX^i+\Lambda^i_jX^j+\Lambda(2,0)X^i \tag{3.3}$$

are the components of a contravariant vector.

Hence the expressions (3.3) give the base connections in K_n^2, a special Kawaguchi space of order 2 and dimension n.

Along the curve, the equations (3.3) become

$$\frac{\overset{\mathrm{I}(1)}{\delta} x^i}{dt} = x''^i + 2H^i + \frac{1}{2}\mu x'^i = x^{|2|i} + \frac{1}{2}\mu x'^i = X^i, \quad \frac{\overset{\mathrm{I}(2)}{\delta} x^i}{dt} = \frac{dX^i}{dt} + H^i_{(j)}X^i + \mu X^i.$$

From $v^i(\bar{t}) = \gamma^h v^i$, we have

$$v^i_{(2)j}x'^j = 0, \quad v^i_{(1)j}x'^j + v^i_{(2)j}x''^j = hv^i.$$

Thus, using these equations and (3.3), we have a covariant derivative, that is,

$$\overset{\mathrm{I}}{\delta} v^i = dx^k \overset{1}{\nabla}_k v^i + \overset{\mathrm{I}}{\delta} x'^k \overset{2}{\nabla}_k v^i + \overset{\mathrm{I}(2)}{\delta} x^i \overset{3}{\nabla}_k v^i,$$

where

$$\overset{1}{\nabla}_k v^i = \frac{\partial v^i}{\partial x^k} + H^i_{(j)(k)}v^j - v^i_{(j)}H^j_{(k)} - kF^{-1}S_{kj}X^j v^i - v^i_{(2)l}\Big(2H^l_k + H^l_{(j)(k)}X^j - 2H^l_{(j)}H^j_{(k)}$$

$$+ \frac{1}{2}H^l_{(k)}\mu\Big),$$

$$\overset{2}{\nabla}_k v^i = v^i_{(k)} + D'^i_{jk}v^j + R_k v^i - v^i_{(2)l}(2H^l_{(k)} + D'^l_{jk}X^j + \mu_{(2)k}X^l) - \frac{1}{2}\mu v^i_{(2)k}$$

$$= \phi_k(v^i) + D'^i_{jk}v^j - v^i_{(2)l}D'^l_{jk}X^j - \frac{1}{2}\mu_{(2)k}v^i\{h-(p-2)k\} - v^i_{(2)l}X^l\mu_{(2)k},$$

$$\overset{3}{\nabla}_k v^i = v^i_{(2)l}\ .$$

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