On Some Gap Theorem for Euler's Method of Summation of Series

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On Some Gap Theorem for Euler's Method of Summation of Series

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三浦白治： 級数のオイラーの総和法に対する或空隙定理に関して（英文）

Hardy and Littlewood (Hardy and Littlewood, Proceedings of the London Mathematical society, (2), vol. 25 (1926)) have proved the following theorem :

For a given series $\sum_{k=1}^{\infty} a_{n_k}$, ($a_{n_k} \neq 0$), let θ be a fixed constant such that

$$\frac{n_{k+1}}{n_k} \geqq \theta > 1, \quad (k=1, 2, \cdots\cdots).$$

If this series be summable by Abel's method of summation to the sum s, then this series is convergent and its sum is s.

Obreschkoff (Obreschkoff, Tôhoku Mathematical Journal, vol. 32 (1930)) obtained also a similar result for Cesàro's method.

Professor Okada (Y. Okada, Bull. of the American Math. Soc. (1937)) obtained the following result for Euler's method :

Theorem. Let $\sum_{n=0}^{\infty} a_n$ be a given series summable by Euler's method, that is, if

$$s_0=0, \quad s_n=a_0+a_1+\cdots\cdots\cdots\cdots+a_{n-1}, \ (n \geqq 1),$$

$$\lim_{n\to\infty} \frac{1}{2^n}\left\{s_0+ns_1+\frac{n(n-1)}{2!}s_2+\cdots\cdots\cdots\cdots+s_n\right\}=s$$

exists; and for two given increasing sequences $\{n_k\}$, $\{n'_k\}$, ($n_k<n'_k$), of positive integers and for a given number α, ($1 \leqq \alpha < 2$), let

$$a_\nu=0, \quad \text{for} \quad n_k<\nu<n'_k, \quad (k=1, 2, \cdots\cdots\cdots\cdots),$$

$$a_n=O(\alpha^n).$$

If $n'_k/n_k \geqq (1+\eta)/(1-\eta)$, ($k=1, 2, \cdots\cdots\cdots\cdots$), for a positive number η such that

$$(1+\eta)\log(1+\eta)+(1-\eta)\log(1-\eta)-2\log\alpha>0,$$

then

$$\lim_{k\to\infty} \sum_{\nu=0}^{n_k} a_\nu=s.$$

Professor Okada's result is about the special case, p=1 in Euler's method of summation to $\sum_{n=0}^{\infty} a_n$:

$$\lim_{n\to\infty} \frac{1}{(q+1)^n}\left\{\binom{n}{0}q^n s_0+\binom{n}{1}q^{n-1}s_1+\cdots\cdots+\binom{n}{n}s_n\right\}=s, \quad (q=2^p-1).$$

Here we will research in the same way about the general case, $p \geqq 1$.

Theorem. Let $\sum_{n=0}^{\infty} a_n$ be a given series summable by Euler's method, that is, if $s_0=0$, $s_n=a_0+a_1+\cdots\cdots+a_{n-1}$, $(n\geqq 1)$, $q=2^p-1$, $(p\geqq 1)$,

$$(1)\qquad \lim_{n\to\infty}\frac{1}{(q+1)^n}\left\{\binom{n}{0}q^n s_0+\binom{n}{1}q^{n-1}s_1+\cdots\cdots+\binom{n}{n}s_n\right\}=s$$

exists; and for two given increasing sequences $\{n_k\}$, $\{n'_k\}$, $(n_k<n'_k)$, of positive integers and for a given number α, let

$$(2)\qquad a_\nu=0,\quad \text{for}\quad n_k<\nu<n'_k,\quad (k=1,2,\cdots\cdots),$$
$$a_n=O(\alpha^n),$$

where $1\leqq\alpha<\dfrac{q+1}{q}$.

If $n'_k/n_k\geqq(1+\eta)/(1-\eta)$, $(k=1,2,\cdots\cdots)$, for a positive number η such that

$$(1+\eta)\log(1+\eta)+(1-\eta)\log(1-\eta)-2\left(\log\left(\frac{2q}{q+1}\right)+\log\alpha\right)>0,$$

then

$$(3)\qquad \lim_{k\to\infty}\sum_{\nu=0}^{n_k}a_\nu=s.$$

Proof. To prove this, we can consider that all n'_k-n_k-1 are even. Then putting

$$n_k+\frac{n'_k-n_k-1}{2}+1=m,$$

from (2) we have

$$a_{m-1}=a_{m-2}=\cdots\cdots=a_{n_k+1}=0,$$

$$a_m=a_{m+1}=\cdots\cdots=a_{n'_k-1}=0.$$

Hence, if we put

$$s_n^{(p)}=\frac{1}{(q+1)^n}\left\{\binom{n}{0}q^n s_0+\binom{n}{1}q^{n-1}s_1+\cdots\cdots+\binom{n}{n}s_n\right\},$$

then we have

$$s_{2m}^{(p)}-s_m=\frac{1}{(q+1)^{2m}}\left\{q^{2m}s_0+2mq^{2m-1}s_1+\frac{2m(2m-1)}{2!}q^{2m-2}s_2+\cdots\cdots+s_{2m}\right\}$$
$$-\frac{1}{(q+1)^{2m}}\left\{q^{2m}s_m+2mq^{2m-1}s_m+\frac{2m(2m-1)}{2!}q^{2m-2}s_m+\cdots\cdots+s_m\right\}$$
$$=\frac{1}{(q+1)^{2m}}\Big\{-(a_0+\cdots\cdots+a_{n_k})q^{2m}-2m(a_1+\cdots\cdots+a_{n_k})q^{2m-1}-\frac{2m(2m-1)}{2!}$$
$$\times(a_2+\cdots\cdots+a_{n_k})q^{2m-2}-\cdots\cdots-\frac{2m(2m-1)\cdots\cdots(2m-n_k+1)}{n_k!}a_{n_k}q^{2m-n_k}$$
$$+\frac{2m(2m-1)\cdots\cdots(2m-n'_k)}{(n'_k+1)!}a_{n'_k}q^{2m-n'_k-1}+\cdots\cdots+(a_{n'_k}+\cdots\cdots+a_{2m-1})\Big\}.$$

Therefore

$$|s_{2m}^{(p)}-s_m|\leqq\left(\frac{q}{q+1}\right)^{2m}\frac{2m(2m-1)\cdots\cdots(2m-n_k+1)}{n_k!}[\{(|a_0|+\cdots\cdots+|a_{n_k}|)$$
$$+(|a_1|+\cdots\cdots+|a_{n_k}|)+\cdots\cdots+|a_{n_k}|\}+\{|a_{n'_k}|+(|a_{n'_k}|+|a_{n'_k+1}|)+\cdots\cdots$$
$$+(|a_{n'_k}|+\cdots\cdots+|a_{2m-1}|)\}].$$

Since from (2) we can find a positive constant M such that $|a_n|<M\alpha^n$, $(n=0,1,2,\cdots\cdots)$, we get

$$|s_{2m}^{(p)}-s_m|<2M\left(\frac{q\alpha}{q+1}\right)^{2m}(n_k+1)^2\frac{2m(2m-1)\cdots\cdots(2m-n_k+1)}{n_k!}$$

$$< 4M \frac{e^{2m \log \alpha'}}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m+\lambda)\Gamma(m-\lambda)},$$

where $\alpha' = \frac{2q\alpha}{q+1}$, $\lambda = m - n_k$.

Let us now put

$$f(m) = \frac{e^{2m \log \alpha'}}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m+\lambda)\Gamma(m-\lambda)} = \frac{1}{2^{2m}} \left(\frac{2q}{q+1}\alpha\right)^{2m} \frac{2m\Gamma(2m)}{\Gamma(m+\lambda)\Gamma(m-\lambda)}$$

or

$$f(m) = \frac{e^{2m \log \alpha'}}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m(1-\delta))\Gamma(m(1+\delta))},$$

where $\lambda = m\delta$, $(0 < \delta = (n'_k - n_k + 1)/(n'_k + n_k + 1) < 1)$.

Then

$$\log f(m) = 2m \log \alpha' - 2m \log 2 + \log(2m)$$
$$+ (2m - \frac{1}{2}) \log(2m) - 2m + O(1)$$
$$- \left\{m(1-\delta) - \frac{1}{2}\right\} \log((1-\delta)m) + (1-\delta)m + O(1)$$
$$- \left\{m(1+\delta) - \frac{1}{2}\right\} \log((1+\delta)m) + (1+\delta)m + O(1)$$
$$= -m\psi(\delta) + \frac{3}{2} \log m + O(1),$$

where

$$\psi(\delta) = (1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta) - 2\log \alpha'.$$

Since from our assumption we have $1 \leqq \alpha < \frac{q+1}{q}$, that is,

$$1 \leqq \frac{2q}{q+1} \leqq \frac{2q}{q+1}\alpha < 2, \quad \text{we get} \quad 1 \leqq \alpha' < 2.$$

A fixed number η_0, $(1 > \eta_0 \geqq 0)$ such that $\psi(\eta_0) = 0$ for the above α' exists necessarily.

Any number η such that $\eta_0 < \eta < 1$ for this η_0 gives $\psi(\eta) > 0$.

When η is so fixed, it follows from (1) that

$$\lim s_m = \lim s_{2m}^{(p)} = s \quad \text{for} \quad 1 > \delta > \eta.$$

From the other assumption $n'_k/n_k \geqq (1+\eta)/(1-\eta)$ to the above η, we have

$$\frac{n'_k - n_k + 1}{n'_k + n_k + 1} > \eta.$$

Consequently $1 > \delta > \eta$ since

$$\delta = (n'_k - n_k + 1)/(n'_k + n_k + 1).$$

Therefore

$$\lim_{k\to\infty} S_{n_k} = \lim_{m\to\infty} s_m = s.$$

Thus our theorem is proved.

Remark. From Theorem and Knopp's theorem (Knopp, Math. Zeitschrift, vol. 15 (1922)) follows immediately Ostrowski's theorem :

Let $f(Z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series whose radius of convergence is 1.

If $a_\nu = 0$ for $n_k < \nu < n'_k$,

and

$$\frac{n'_k}{n_k} < 1 + \theta, \quad (k = 1, 2, \cdots\cdots),$$

θ being a positive constant, then the partial sums s_{n_k} of this series converge uniformly in a full neighbourhood of every regular point of the function, $f(Z)$ on the unit circle.