接続の和に関する一注意

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A Note on the Sum of Connections

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Summary

Let ω and ω' be two connection forms on a principal bundle P, f a C^∞ function on the base manifold M, and π the projection from P to M. Then it is known that $\omega_f=(f\pi)\omega+(1-f\pi)\omega'$ is also a connection form on P and so is $\omega_a=a\omega+(1-a)\omega'$, where a is constant. We will investigate the relation between the two old connections and the new one, in particular, between their curvature forms. We refer the reader to Kobayashi and Nomizu [2] for the notations used in this paper.

We begin with the lemma which is an answer to Problem 2 of Chapter 5 of Bishop and Crittenden [1].

Lemma 1. *If $X\in T_u(P)$ and $X=hX+X_1$, $X=h'X+X'_1$ are the horizontal and vertical decompositions of X with respect to ω and ω', then X is decomposed with respect to ω_f as follows:*

$$X=\{(f\pi)hX+(1-f\pi)h'X\}+\{(f\pi)X_1+(1-f\pi)X'_1\}$$

Proof. It suffices to prove that the first bracket is horizontal with respect to ω_f. Their exist such A_1 and $A'_1\in\mathfrak{g}$ that $X_1=A_1^*$ and $X'_1=A_1'^*$ *at* $u\in P$. This gives $A_1=\omega(h'X)+A'_1$, $A'_1=\omega'(hX)+A_1$, and therefore

$$\omega(h'X)+\omega'(hX)=0.$$

From this and by the definition of ω_f, we have

$$\omega_f\{(f\pi)hX+(1-f\pi)h'X\}=0.$$

In particular, the decomposition of X with respect to ω_a is

$$X=\{ahX+(1-a)h'X\}+\{aX_1+(1-a)X'_1\}.$$

Notice that the horizontal component with respect to ω_a is also written in the form of $hX+(1-a)(X_1-X'_1)$ or $h'X+a(X'_1-X_1)$.

Theorem 1. *Let Ω, Ω' and Ω_a be the curvature forms of ω, ω' and ω_a respectively. Then*

$$\Omega_a=a\Omega+(1-a)\Omega'-\tfrac{1}{2}a(1-a)[\tau,\tau],$$

where $\tau=\omega-\omega'$, which is called "the difference form."

Proof. This theorem is immediately derived from the structural equations of the three connections.

$$\Omega_a = d\omega_a + \tfrac{1}{2}[\omega_a, \omega_a]$$
$$= a(\Omega - \tfrac{1}{2}[\omega, \omega]) + (1-a)(\Omega' - \tfrac{1}{2}[\omega', \omega']) + \tfrac{1}{2}[a\omega + (1-a)\omega', a\omega + (1-a)\omega']$$
$$= a\Omega + (1-a)\Omega' - \tfrac{1}{2}a(1-a)[\omega - \omega', \omega - \omega']$$

It is clear that the above theorem holds for the connection ω_f with a little modification, that is, if Ω_f is the curvature form of ω_f, then

$$\Omega_f = (f\pi)\Omega + (1-f\pi)\Omega' - \tfrac{1}{2}(f\pi)(1-f\pi)[\tau, \tau] + d(f\pi) \wedge \tau.$$

Lemma 2. *$d\Omega(X, Y, Z) = \frac{1}{3}C\{[\Omega(X, Y), \omega(Z)]\}$, $d[\tau, \tau](X, Y, Z) = \frac{1}{3}C\{[\tau(X), \tau(Y), \omega(Z)]\} + D[\tau, \tau](X, Y, Z)$ for X, Y and $Z \in T_u(P)$, where we denote by C the cyclic sum of X, Y and Z.*

Proof. If ϕ is a $\mathfrak{g}$-valued horizontal 2-form of type adG and D as the exterior covariant differentiation with respect to ω, then the following formula is found (see p. 86 of [1]):

$$d\phi(X, Y, Z) = \tfrac{1}{3}C\{[\phi(X, Y), \omega(Z)]\} + D\phi(X, Y, Z)$$

To verify the two equations we have only to take Ω or $[\tau, \tau]$ as ϕ, and apply Bianchi's identity.

Now we apply the exterior covariant differentiation D_a with respect to ω_a to the formula in Theorem 1. By Bianchi's identity we have

$$aD_a\Omega + (1-a)D_a\Omega' - \tfrac{1}{2}a(1-a)D_a[\tau, \tau] = 0$$

Noticing the remark which is subsequent to Lemma 1, for X, Y and $Z \in T_u(P)$

$$D_a\Omega(X, Y, Z) = d\Omega\{hX + (1-a)(X_1 - X_1'), hY + (1-a)(Y_1 - Y_1'), hZ + (1-a)(Z_1 - Z_1')\}$$
$$= (1-a)\{d\Omega(X_1 - X_1', hY, hZ) + d\Omega(hX, Y_1 - Y_1', hZ) + d\Omega(hX, hY, Z_1 - Z_1')\}$$
$$= \tfrac{1}{3}(1-a)C\{[\Omega(X, Y), \tau(Z)]\}$$

because $d\Omega$ vanishes by Lemma 2 if two are vertical and by Bianchi,s identity if all are horizontal with respect to ω. And similarly we have

$$D_a\Omega' = -\tfrac{1}{3}aC\{[\Omega'(X, Y), \tau(Z)].$$

Therefore, when a is neither 0 nor 1, we have

$$D_a[\tau, \tau](X, Y, Z) = \tfrac{2}{3}C\{[(\Omega - \Omega_1)(X, Y), \tau(Z)]\}.$$

This formula means that the exterior covariant derivative of $[\tau, \tau]$ with respect to ω_a is independent of the value of a except 0 and 1. But the exception can be removed by the following lemma.

Lemma 3. *$D_a[\tau, \tau] = D_1[\tau, \tau]$ for every a where D_1 is the exterior covariant differentiation with respect to ω.*

Proof. For X, Y and $Z \in T_u(P)$,

$$D_a[\tau, \tau](X, Y, Z) = d[\tau, \tau]\{hX + (1-a)(X_1 - X_1'), hY + (1-a)(Y_1 - Y_1'), hZ + (1-a)(Z_1 - Z_1')\}.$$

Here, notice that $d[\tau, \tau](hX, hY, hZ) = D_1[\tau, \tau](X, Y, Z)$, $d[\tau, \tau](hX, hY, Z_1 - Z_1') = \frac{1}{3}\{[\tau(X), \tau(Y)], \tau(Z)\}$ and so on. As other kinds of terms vanish by Lemma 2, we have

$$D_a[\tau, \tau](X, Y, Z) = D_1[\tau, \tau](X, Y, Z) + \tfrac{1}{3}(1-a)C\{[[\tau(X), \tau(Y)], \tau(Z)]\}$$
$$= D_1[\tau, \tau](X, Y, Z)$$

because the second term vanishes by Jacobi's identity.

By Lemma 3 and the preceding argument, we have the following conclusion.

Theorem 2. *For every a and X, Y and* $Z \in T_u(P)$

$$D_a[\tau, \tau](X, Y, Z) = \tfrac{2}{3}C\{[(\Omega - \Omega')(X, Y), \tau(Z)].$$

REFERENCES

[1] Bishop, R. and Crittenden, R. (1964), Geometry oj Manifolds. Academic Press, New York and London.

[2] Kobayashi, S. and Nomizu, K. (1963), Foundations of Differential Geometry. Vol. I. Wiley (Interscience), New York.