超曲面のオイラー・ポアンカレ特性類について

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Remarks on the Euler-Poincaré Characteristic of a Hypersurface

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§ 1. Preliminaries.

1. Generalized Gauss-Bonnet formula.

Let M be a compact orientable Riemannian manifold of an even dimension $n(=2m)$ throughout this paper. The Euler-Poincaré characteristic $\chi(M)$ of M is given by

(1) $$\chi(M)=(2/c_n)\int_M K_n dV,$$

where c_n is the volume of the Euclidian unit n-sphere, K_n donetes the Lipschitz-Killing curvature of M and dV is the volume element of M. This formula is called the generalized Gauss-Bonnet formula ([2], [3], [4]).

2. The Gauss equation for a hypersurface.

Let $\bar{M}$ be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V\,;\,x^\lambda\}$ and $g_{\lambda\mu}$ and $\bar{R}_{\lambda\mu\nu\omega}$, the metric tensor and the curvature tensor respetively.

Let M be covered by a system of coordinate neighborhoods $\{U\,;\,u^a\}$ and g_{ab} and R_{abcd}, the metric tensor and curvature tensor of M respectively. Let M be immersed in $\bar{M}$ and

$$x^\lambda=x^\lambda(u^a)$$

be the local parametric expression of M.

Throughout this paper, Greek indices run over the range $\{1, 2, \dots, n+1\}$ and Latin indices the range $\{1, 2, \dots, n\}$.

If we put

(2) $$B_a^\lambda=\partial_a x^\lambda,\ \partial_a=\partial/\partial u^a,$$

then, the Riemannian metric of M induced from that of $\bar{M}$ is given by

(3) $$g_{ab}=\bar{g}_{\lambda\mu}B_a^\lambda B_b^\mu,$$

and the equations of Gauss are presented by

(4) $$R_{abcd}=\bar{R}_{\lambda\mu\nu\omega}B_a^\lambda B_b^\mu B_c^\nu B^\omega-H_{ac}H_{bd}+H_{ad}H_{bc},$$

where H_{ab} are the components of the second fundamental tensor H and $H_{ab}=H_{ba}$.

§ 2. Some results.

Theorem 1. *Let M be a hypersurface of a space of constant curvature $c\geqq 0$ (resp. $c\leqq 0$). If the second curvature tensor is always positive (resp. negative), then $\chi(M)$ (resp.*

$(-1)^m\chi(M))$ *is non-negative.*

Proof. Because $\bar{M}$ is a space of constant curvature,

(5) $$\bar{R}_{\lambda\mu\nu\omega}=-c(\bar{g}_{\lambda\nu}\bar{g}_{\mu\omega}-\bar{g}_{\lambda\omega}\bar{g}_{\mu\nu}).$$

From (3), (4) and (5), we have

(6) $$R_{abcd}=-c(g_{ac}g_{bd}-g_{ad}g_{bc})-(H_{ac}H_{bd}-H_{ad}H_{bc}).$$

We can choose the orthonormal coordinate system for a tangent space of M so that $H_{ab}=0(a\neq b)$ at any point of M.

With respect to this coodinate system,

(7) $$H=\begin{pmatrix} k_1 & & 0 \\ & k_2 . & \\ & & \ddots \\ 0 & & k_n \end{pmatrix}$$

where $k_1, k_2, \ldots, k_n$ are the eigenvalues of H.

If a, b, c, d are different mutually,

(8) $$\begin{cases} -R_{abab}=c+k_a k_b, \\ R_{abac}=0 \text{ and } R_{abcd}=0. \end{cases}$$

If, in particular, H is positive (resp. negative) and $c\geqq 0$ (resp. $c\leqq 0$),

$$-R_{abab}\geqq 0 \text{ (resp. } -R_{abab}\leqq 0).$$

In this case,

(9) $$(-1)^m K_n=\frac{1}{2^m n!}\varepsilon^{i_1\cdots i_n}\varepsilon^{j_1\cdots j_n}R_{i_1 i_2 j_1 j_2}\cdots R_{i_{n-1}i_n j_{n-1}j_n}\geqq 0.$$

Comparison with (1) completes the proof. Q.E.D.

Corollary. *Let M be a hypersurface of a space of constant curvature. If the sectional curvature of M is always non-negative (resp. non-negative), then $\chi(M)$ (resp. $(-1)^m\chi(M)$) is non-negative.*

Similarly we have

Theorem 2. *Let M be a hypersurface of a conformally flat space $\bar{M}$. If the sectional curvature of M is always non-negative (resp. nonpositive) and*

$$\bar{R}_{\lambda\mu}B_a^\lambda B_b^\mu=\alpha g_{ab}+\beta H_{ab},$$

where α and β are the funtcions on M, then $\chi(M)$ (resp. $(-1)^m\chi(M)$) is non-negative.

Remark 1. *If* $\beta\equiv 0$ in the Theorem 2, then $\bar{M}$ is an Einstein space. In this case $\bar{M}$ is the space of constant curvature consequently.

Remark 2. It is sufficient that a neighborhood of any point of M is immersed in $\bar{M}$ in these theorems.

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