

超曲面のオイラー・ポアンカレ特性類について

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Remarks on the Euler-Poincaré Characteristic of a Hypersurface

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§ 1. Preliminaries.

1. Generalized Gauss-Bonnet formula.

Let M be a compact orientable Riemannian manifold of an even dimension $n (= 2m)$ throughout this paper. The Euler-Poincaré characteristic $\chi(M)$ of M is given by

$$(1) \quad \chi(M) = (2/c_n) \int_M K_n dV,$$

where c_n is the volume of the Euclidian unit n -sphere, K_n denotes the Lipschitz-Killing curvature of M and dV is the volume element of M . This formula is called the generalized Gauss-Bonnet formula ([2], [3], [4]).

2. The Gauss equation for a hypersurface.

Let \bar{M} be an $(n+1)$ -dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; x^i\}$ and $g_{\lambda\mu}$ and $\bar{R}_{\lambda\mu\nu\omega}$, the metric tensor and the curvature tensor respectively.

Let M be covered by a system of coordinate neighborhoods $\{U; u^a\}$ and g_{ab} and R_{abcd} , the metric tensor and curvature tensor of M respectively. Let M be immersed in \bar{M} and

$$x^i = x^i(u^a)$$

be the local parametric expression of M .

Throughout this paper, Greek indices run over the range $\{1, 2, \dots, n+1\}$ and Latin indices the range $\{1, 2, \dots, n\}$.

If we put

$$(2) \quad B_a^\lambda = \partial_a x^\lambda, \quad \partial_a = \partial / \partial u^a,$$

then, the Riemannian metric of M induced from that of \bar{M} is given by

$$(3) \quad g_{ab} = \bar{g}_{\lambda\mu} B_a^\lambda B_b^\mu,$$

and the equations of Gauss are presented by

$$(4) \quad R_{abca} = \bar{R}_{\lambda\mu\nu\omega} B_a^\lambda B_b^\mu B_c^\nu B_a^\omega - H_{ac} H_{bd} + H_{ad} H_{bc},$$

where H_{ab} are the components of the second fundamental tensor H and $H_{ab} = H_{ba}$.

§ 2. Some results.

Theorem 1. *Let M be a hypersurface of a space of constant curvature $c \geq 0$ (resp. $c \leq 0$). If the second curvature tensor is always positive (resp. negative), then $\chi(M)$ (resp.*

$(-1)^m \chi(M)$ is non-negative.

Proof. Because \bar{M} is a space of constant curvature,

$$(5) \quad \bar{R}_{\lambda\mu\nu\omega} = -c(\bar{g}_{\lambda\nu}\bar{g}_{\mu\omega} - \bar{g}_{\lambda\omega}\bar{g}_{\mu\nu}).$$

From (3), (4) and (5), we have

$$(6) \quad R_{abcd} = -c(g_{ac}g_{bd} - g_{ad}g_{bc}) - (H_{ac}H_{bd} - H_{ad}H_{bc}).$$

We can choose the orthonormal coordinate system for a tangent space of M so that $H_{ab} = 0 (a \neq b)$ at any point of M .

With respect to this coordinate system,

$$(7) \quad H = \begin{pmatrix} k_1 & & 0 \\ & k_2 & \\ & & \ddots \\ 0 & & & k_n \end{pmatrix}$$

where k_1, k_2, \dots, k_n are the eigenvalues of H .

If a, b, c, d are different mutually,

$$(8) \quad \begin{cases} -R_{abab} = c + k_a k_b, \\ R_{abac} = 0 \text{ and } R_{abcd} = 0. \end{cases}$$

If, in particular, H is positive (resp. negative) and $c \geq 0$ (resp. $c \leq 0$),

$$-R_{abab} \geq 0 \text{ (resp. } -R_{abab} \leq 0).$$

In this case,

$$(9) \quad (-1)^m K_n = \frac{1}{2^m n!} \epsilon^{i_1 \dots i_n} \epsilon^{j_1 \dots j_n} R_{i_1 i_2 j_1 j_2 \dots} R_{i_{n-1} i_n j_{n-1} j_n} \geq 0.$$

Comparison with (1) completes the proof. Q.E.D.

Corollary. Let M be a hypersurface of a space of constant curvature. If the sectional curvature of M is always non-negative (resp. non-negative), then $\chi(M)$ (resp. $(-1)^m \chi(M)$) is non-negative.

Similarly we have

Theorem 2. Let M be a hypersurface of a conformally flat space \bar{M} . If the sectional curvature of M is always non-negative (resp. nonpositive) and

$$\bar{R}_{\lambda\mu} B_a^\lambda B_b^\mu = \alpha g_{ab} + \beta H_{ab},$$

where α and β are the functions on M , then $\chi(M)$ (resp. $(-1)^m \chi(M)$) is non-negative.

Remark 1. If $\beta \equiv 0$ in the Theorem 2, then \bar{M} is an Einstein space. In this case \bar{M} is the space of constant curvature consequently.

Remark 2. It is sufficient that a neighborhood of any point of M is immersed in \bar{M} in these theorems.

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