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メタデータ	言語: English 出版者: 北海道学芸大学 公開日: 2012-11-07 キーワード: 作成者: 矢野, 晋平 メールアドレス: 所属:
URL	https://doi.org/10.32150/00000213

Vol. 5, No. 1 Journal of Hokkaido Gakugei University Feb. 1954

ON THE EUCLIDEAN CONNECTION IN AN AREAL SPACE OF GENERAL TYPE (I).

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矢野晋平 : 面積空間における Euclidean Connection について

§ 0. Introduction.

If an m-dimensional subspace in an n-dimensional space is given in the parameter forms :

$$(0.1) \qquad x^i = x^i(u^\alpha), \qquad i = 1, 2, \cdots, n, \qquad \alpha = 1, 2, \cdots, m,$$

then the area of a domain D on the subspace is defined by the integral

$$(0.2) \qquad \int \cdots_D \int F(x^i, \frac{\partial x^i}{\partial u^\alpha}) d(u^1, \cdots, u^m),$$

where F is the *a priori* given function. Such space is called an *areal space* and the theory was treated at first by A. KAWAGUCHI and S. HOKARI [8][1], and its geometry has been discussed by several writers [4]—[18]. In the space, the difficulty is that, even a value of a function depending on the plane elements $p^{i_1 \cdots i_m}$ is fixed but the values of the partial derivatives by $p^{i_1 \cdots i_m}$ are not fixed, because the plane elements $p^{i_1 \cdots i_m}$ are not independent and there are Plüker's identities among them, i. e. $p^{[i_1 \cdots i_m} p^{j_1] \cdots j_m} = 0$. In this view there are two ways to study the theory ; one of them is to substitute the partial derivatives by $p^{i_1 \cdots i_m}$ for the ***intrinsic derivatives*** [12], [13] and the other is to use the mutually independent parameters p^i_α in place of $p^{i_1 \cdots i_m}$ [14]—[17]. We adopt the first way in the present paper.

The theorys of connection were descused by the previous writers, in the main, in the case of the submetric class. In the present paper we discuss the *Euclidean connection* in general type, ***not submetric class***, according to the Finsler and Cartan spaces [1]—[3] with $m=2$. In an areal space, one of the most important problems is to determine the metric bitensor $g_{ij,kl}$ [8], [9], [11] and other one is the connection parameters [10], [12], [14], in the present paper we discuss too these problems in another way. The relations of the fundamental functions to the intrinsic derivative are expressed in §2, §3. In §4 the connection parameters C and the base connection are determined as the method in the Finsler and Cartan spaces [1]—[3]. The ralations between the covariant derivatives

1) Numbers in brackets refer to the references at the end of the paper.

of a tensor and the curvature tensors of the space are discussed in §5 and §6.

§ 1. Fundamental equations.

We confine ourselves to the case that the dimension of the subspace is two, i.e. $m=2$, in (0.1). From the theory of integral invariant the fundamental function F can be rewritten into a function F depending on only the variables x^i and p^{ij}, where $p^{ij}=2\frac{\partial x^i}{\partial u^{[1}}\frac{\partial x^j}{\partial u^{2]}}$, and must be a P-scalar density of weight -1, and positively homogeneous degree one in p^{ij} [1], i.e.

$$F(x'^i,\ p'^{ij})=\left(\frac{\partial(u'^1u'^2)}{\partial(u^1,\ u^2)}\right)^{-1}F(x^i,\ p^{ij}),\qquad \frac{\partial F}{\partial p^{ij}}p^{ij}=2F,\qquad \frac{\partial^2 F}{\partial p^{ij}\partial p^{kl}}p^{kl}=0,$$

and the function $L\equiv\frac{1}{2}F^2$ is a P-scalar densty of weight -2, and homogeneous of degree two in p^{ij}, i.e.

$$(1.1)\qquad \frac{\partial L}{\partial p^{ij}}p^{ij}=4L,\qquad \frac{\partial^2 L}{\partial p^{ij}\partial p^{kl}}p^{ij}p^{kl}=8L.$$

It is evident that the bivector p^{ij} is simple and a relative P-scalar of weight -1 and Plücker's identities hold good, i.e.

$$(1.2)\qquad P^{ijkl}\equiv 3p^{[ij}p^{kl]}=p^{ij}p^{kl}+p^{ik}p^{lj}+p^{il}p^{jk}=0.$$

We show at first a frequently useful theorem in the present paper.

Theorem [1.1]. *If A_{ij} is skew-symmetry with respect to i and j, i.e. $A_{(ij)}=0$, then we have*

$$A_{ij}p^{i\alpha}p^{j\beta}=\frac{1}{2}A_{ij}p^{ij}p^{\alpha\beta}.$$

Proof : From (1.2) we have following relations proving the Theorem,

$$A_{ij}p^{i\alpha}p^{j\beta}=\frac{1}{2}(A_{ij}-A_{ji})p^{i\alpha}p^{j\beta}=\frac{1}{2}A_{ij}(p^{i\alpha}p^{j\beta}-p^{i\beta}p^{j\alpha})=\frac{1}{2}A_{ij}p^{ij}p^{\alpha\beta}.$$

The transversal bivector $G_{ij}\equiv\frac{1}{2}L^{-1}(L_{i\alpha}L_{j\beta}p^{\alpha\beta})$ [2] is simple and its components satisfy the relation $G_{ij}p^{ij}=4L$, and in virtue of Theorem [1.1] we have the relations

$$(1.3)\qquad 3G_{[ij}G_{kl]}=G_{ij}G_{kl}+G_{ik}G_{lj}+G_{il}G_{jk}=0,$$

$$(1.4)\qquad G_{ij}p^{i\alpha}p^{j\beta}=\frac{1}{2}(G_{ij}p^{ij}p^{\alpha\beta})=2Lp^{\alpha\beta}.$$

§ 2. The intrinsic derivatives by p^{ij},

Let us consider a function $W(x,p)$ being homogeneous degree ρ in p^{ij} and weight σ. The variables p^{ij} are not independent of each other and there are relations (1.2) and

1) See [11], p. 19.
2) See [11], p 22.

$p^{(ij)}=0$ among them, consequently, even if the functional value of $W(x,p)$ is fixed the functional forms are indeterminate, so the partial derivatives of $W(x,p)$ by p^{ij} have not fixed value, but as usual the theory of the Finsler and Cartan spaces[1], we can easily see

Theorem [2. 1]. *The partial derivatives of $W(x,p)$ by p^{ij} are P-scalar density of weight $\sigma+1$ and positively homogeneous of degree $\rho-1$ in p^{ij}, and is a covariant tensor degree two under coordinate transformations.*

In order to determine the values of partial derivatives of $W(x,p)$ by p, A. KAWAGUCHI introduced the *intrinsic derivatives*[2], in the sign ; , operating $W(x,p)$ in the form

$$(2.1)\qquad W_{;ij}\equiv\frac{1}{L}\frac{\partial W}{\partial p_{[l|k|}}G_{j]l}p^{kl}-\frac{\rho W}{2L}G_{ij},$$

among them there are relatios

$$(2.2)\qquad W_{;ij}p^{ij}=2\rho W,\qquad\qquad W_{;[ij}G_{kl]}=0.$$

It must be noticed that, even if any two functions $W(p)$ and $W'(p)$ have a same value in the manifold of Grassmann, i, e. $W\equiv W'$ $(mod\ C_2)$[3], their partial derivatives by p^{ij} do not always have the same value, but their intrinsic derivatives have a same value in the manifold of Grassmann, i. e. $W_{;ij}\equiv W'_{;ij}$ $(mod\ G_2)$.

L and G_{ij} are P-scalar densities of weight -2 and -1 and positively homogeneous of degree 2 and 1 respectivly, from Theorem [2. 1] we have

Theorem [2. 2]. *Let $W(x,p)$ be a P-scalar density of weight σ and positively homogeneous of degree ρ in p, then $W_{;ij}$ is a covariant bivector of weight $\sigma+1$ and positively homogeneous of degree $\rho-1$ in p and the quantities $W_{;ij}$ are independent of the functional forms of W and uniquely determined from any functional form of W.*

In order to seek the characters of the intrinsic derivatives, let $V(x,p)$ be any function of positively homogeneous of degree τ in p, then the function $W\cdot V$ is positively homogeneous of degree $\rho+\tau$ in p, (2.1) leads us to $(W\cdot V)_{;ij}=W_{;ij}\cdot V+W\cdot V_{;ij}$, and we have

Theorem [2. 3]. *The intrinsic derivatives of outer and inner multiplication of tensors and functions obeys the same rules as in ordinary differentiation.*

The bivector $W_{;ij}$ is skew-symmetry with respect to the indices i and j, from Theorem [1. 1] and the first equation of (2.2) we have

$$(2.3)\qquad W_{;\alpha\beta}p^{\alpha\lambda}p^{\beta\mu}=\frac{1}{2}W_{;\alpha\beta}p^{\alpha\beta}p^{\lambda\mu}=\rho Wp^{\lambda\mu},$$

moreover (1.4) and (2.3) lead us to

$$(2.4)\qquad W_{;i\alpha}p^{\alpha\beta}=\partial W/\partial p^{i\alpha}\cdot p^{\alpha\beta}.$$

Differentiating the identity $3p^{[\rho\sigma}p^{ij]}=0$ and multiplying by $G_{j\sigma}$ successively and

1) See [1]-[3].
2) See [12], p 77.
3) See [12], p 77, Theorem [2. 7].

summing with respect to the index j, we have $G_{j\sigma}p^{\rho\sigma}dp^{ij}=2G_{j\sigma}p^{\rho(i}dp^{|\sigma j)}-4Ldp^{\rho i}-2G_{j\sigma}\cdot dp^{\rho\sigma}p^{ij}$, putting this equation in the first term of right-hand side of following relation $W;_{ij}dp^{ij}=(L^{-1}\cdot\partial W/\partial p^{i\rho}\cdot G_{j\sigma}p^{\rho\sigma}-\rho W/2L\cdot G_{ij})dp^{ij}$, from (2.1) and (2.2) after simple calculations we have

$$(2.5)\qquad W;_{ij}dp^{ij}=\partial W/\partial p^{ij}\cdot dp^{ij}.$$

The last equation means that, in the general case, we can not conclude the relations $W;_{ij}=\partial W/\partial p^{ij}$ but the differential dW of W is given by

$$(2.6)\qquad dW=\partial W/\partial x^i\cdot dx^i+\frac{1}{2}W;_{ij}dp^{ij}.$$

The relations (2.5) and (2.6) were finded by A. KAWAGUCHI and K. TANDAI in another way[1].

§ 3. The derivatives of fundamental functions.

In this Chapter we shall show the relations between the fundamental functions and their intrinsic derivatives. At first considering the relation $\partial p^{ij}/\partial p^{\rho\sigma}=\delta^i_\rho\delta^j_\sigma+\delta^i_\sigma\delta^j_\rho$ [2] and (2.1) we have

$$(3.1)\qquad P^{ij};_{kl}=L^{-1}(2\delta^{(i}_{[k}G_{l]\sigma}p^{j)\sigma}-\frac{1}{2}p^{ij}G_{kl})^{3)}.$$

From the definition of G_{jl} and Theorem [1.1] we have the relations $G_{jl}p^{kl}=\frac{1}{2}L^{-1}L_{j\alpha}\cdot L_{l\beta}l^{\alpha\beta}l^{kl}=L_{j\alpha}p^{k\alpha}$, where $l^{ij}\equiv p^{ij}F^{-1}$, and L is homogeneous of degree two in P, consequently we have $L;_{ij}=G_{ij}$, i. e.

Theorem [3.1]. *$L;_{ij}$ is the transversal bivector G_{ij}.*

From Theorem [3.1] sinc $L;_{ij}$ is homogeneous of degree one in p, we have

$$(3.2)\qquad L;_{ij};_{kl}=-\frac{1}{2}L^{-1}(G_{ij}G_{kl}+X_{ij,kl}),$$

where : $X_{ij,kl}=2G_{ij,[k|\rho}G_{\sigma,l]}p^{\rho\sigma}$, $G_{ij,k\rho}\equiv\partial G_{ij}/\partial p^{k\rho}$,

From the epuation $(G_{ij,k\nu}G_{l}+G_{ij,l}G_{k\nu})p^{\nu}=(G_{kl,i\nu}G_{\eta j}+G_{kl,j}G_{i\nu})p^{\nu\eta}$ explained by A. KAWAGUCHI[4] and the definition of $X_{ij,kl}$ we have

$$(3.3)\qquad X_{ij,kl}=X_{kl,ij}=-X_{ji,kl}=-X_{ij,lk},$$

consequently, from (3,2) we have

$$(3.4)\qquad L;_{ij};_{kl}=L;_{kl};_{ij}=-L;_{ji};_{kl}=-L;_{ij};_{lk}.$$

From Theorem [2.2] and [3.4] we have

Theorem [3.2] *$L;_{ij;kl}$ is a symmetric bitensor of weight zero and degree of zero in*

1) See [16] p 51.
2) See [11] p 18.
3) See [12] p 78.
4) See [11] p 25.

p, i. e, intrinsic.

However, we can not conclude the relations $L_{;ij;kl}=g^*_{ij,kl}$ where $g^*_{ij,kl}$ is metric bitensor[1], because, from the second equation of (2.2) the relations $L_{;ij;[kl}G_{rs]}=0$ hold good but $g^*_{ij,[kl}G_{rs]}$ may not be equal to zero. We shall compute later on the difference between them. At first (1.3) may be deformed in the form

$$(3.5) \qquad 2G_{i[j}G_{k]l}=G_{il}G_{kj},$$

multiplyng both sides of (3.5) by p^{jk} and contracting the indices j and k we have

$$(3.6) \qquad G_{ij}G_{lk}p^{jk}=2LG_{il}.$$

Because of (2.4), equations (2.1) reduce to

$$(3.7) \qquad W_{;[i|\rho|}G_{j]\sigma}p^{\rho\sigma} = LW_{;ij}+\frac{1}{2}(\rho W)G_{ij}.$$

Taking the intrinsic derivatives of both sides of (3.7) and considering (3.1), (3.5), (3.6) and (3.7) we have

$$(W_{;[i|\rho|}G_{j]\sigma})_{;kl}p^{\rho\sigma}=G_{kl}W_{;ij}+LW_{;ij;kl}+\frac{1}{2}\rho(W_{;kl}G_{ij}+WG_{ij;kl})-W_{;[i\,\rho|}G_{j]\sigma}p^{\rho\sigma}{}_{;kl}$$
$$=W_{;ij}G_{kl}+LW_{;ij;kl}+\frac{1}{2}\rho(W_{;kl}G_{ij}+WG_{;ij;kl})-2W_{;[i[k}G_{j]l]},$$

from the above equations and (3.4) we have

$$(3.8) \qquad W_{;ij;kl}-W_{;kl;ij}=L^{-1}\{(W_{;[i|\rho}G_{j]\rho})_{;kl}p^{\rho\sigma}+\frac{1}{2}(\rho-2)W_{;ij}G_{kl}\}$$
$$-cycl\ (ik),\ (jl)=L^{-1}\{W_{;[i\,\rho;kl|}G_{j]\sigma}p^{\rho\sigma}+W_{;[i|\rho|}G_{;j]\sigma;kl}p^{\rho\sigma}$$
$$+\frac{1}{2}(\rho-2)W_{;ij}G_{kl}\}-cycl\ (ik),\ (jl),$$

where the sign *cycl* (ik), (jl) means the terms which given by to exchange i for k and j for l in the all of former terms.

On the other hand from (2.4), considering the definition of the intrinsic derivatives we have $W_{;i\rho;kl}\,p^{\rho\sigma}+W_{;i\rho}\,p^{\rho\sigma}{}_{;kl}=L^{-1}\left\{\frac{\partial(\partial W/\partial p^{i\rho}\cdot p^{\rho\sigma})}{\partial p^{[k|\alpha|}}\,G_{l]\beta}p^{\alpha\beta}-\frac{1}{2}\rho\,W_{;i\beta}\,p^{\beta\sigma}G^{kl}\right\}$, multiplying both sides of this equation by $G_{j\sigma}$ and summing with σ and taking the symmetric part with respect to i and j, and considering (3.1), (3.5), (3.6) and (3.7), after some calculations we have

$$W_{;[i|\rho;kl|}G_{j]\sigma}p^{\rho\sigma}+2W_{;[i[k}G_{j]l]}-cycl\ (ik),\ (jl)=L^{-1}\left\{\frac{\partial^2 W}{\partial p^{[i|\rho|}\partial p^{[k|\alpha|}}\,G_{j]\sigma}G_{l]\beta}p^{\rho\sigma}p^{\alpha\beta}\right.$$
$$\left.+2L\,\frac{\partial W}{\partial p^{[i[k}}\,G_{j]l]}+\frac{1}{2}L(1-\rho)(LW_{;ij}+\frac{1}{2}\rho WG_{ij})G_{kl}\right\}-cycl\ (ik),\ (jl),$$

that is $W_{;[i\,\rho;kl|}\,G_{j]\sigma}\,p^{\rho\sigma}-cycl\ (ik),\ (jl)=\frac{1}{2}(1-\rho)\,W_{;ij}G_{kl}-cycl\ (ik),\ (jl)$, putting the last relation in the first term of right-hand side of (3.8) we have finally the relations

1) See [*11*], p *25*, (3.16). $g^*_{ij,kl}$ means $g_{ij,kl}$ in [11].

of the commutation of the intrinsic derivatives with W

$$(3.9)\qquad W_{;ij;kl}-W_{;kl;ij}=L^{-1}\{W_{,[i|\rho|}G_{;j]\sigma;k}\,p^{\rho\sigma}-\frac{1}{2}W_{;ij}G_{kl}\}-cycl\ (ik),\ (jl).$$

Hence we have

Theorem [3.3]. *The intrinsic derivative is not commutative in general.*

However, if the both sides of (3.9) be multipled by p^{ij} and contracted with i and j, because of (2.3) and (3.7), the right-hand side of the member vaniches, and from Theorem [2.2] we have

$$(3.10)\qquad W_{;ij;kl}p^{ij}=W_{;kl;ij}p^{ij}=2(\rho-1)W_{;kl}.$$

The metric bitensor $g^{*}_{ik,kl}$ may be represented by our notations in the form

$$(3.11)\qquad g^{*}_{ij,kl}=-\frac{1}{4}L^{-1}(2G_{ij}G_{kl}+X_{ij,kl}-\Lambda_{ij,kl}),$$

where, $\Lambda_{ij,kl}\equiv 2G_{i\rho,[j\alpha}G_{k\sigma,l]\beta}p^{\rho\sigma}p^{\alpha\beta}$.

Even $\Lambda_{ij,kl}$ satisfis the same relations as $X_{ij,kl}$, i. e.

$$(3.12)\qquad \Lambda_{ij,kl}=\Lambda_{kl,ij}=-\Lambda_{jl,kl}=-\Lambda_{ij,lk},$$

we must not conclude that $\Lambda_{ij,kl}=-X_{ij,kl}$. Because, from (3.2) and (3.11) we have

$$(3.13)\qquad g^{*}_{ij,kl}-L_{;ij;kl}=\frac{1}{4}L^{-1}(\Lambda_{ij,kl}+X_{ij,kl}),$$

and the left-hand sides of (3.13) do not vanich.

From the definitions of $X_{ij,kl}$ and $\Lambda_{ij,kl}$, and the relations $G_{i\rho,j\sigma}p^{\rho\alpha}p^{i\beta}=G_{j\sigma}p^{\beta\alpha}$ we have

$$(3.14)\qquad \Lambda_{ij,kl}p^{il}=-X_{ij,kl}p^{il},$$

consequently, (3.13) leads us to

$$(3.15)\qquad g^{*}_{ij,kl}p^{il}=L_{;ij;kl}p^{il}.$$

From (3.2) and (3.11), $\Lambda_{ij,kl}$ and $X_{ij,kl}$ may be solved in the form

$$(3.16)\qquad X_{ij,kl}=-(G_{ij}G_{kl}+2LL_{;ij;kl}),\quad \Lambda_{ij,kl}=4Lg^{*}_{ij,kl}-2LL_{;ij;kl}+G_{ij}G_{kl}.$$

From Theorem [3.2], and (3.15) we have

Theorem [3.4]. *The symmetric bitensor $L_{;ij;kl}$ is not equal to the metric bitensor $g^{*}_{ij,kl}$, but we may adopt $L_{;ij;kl}$ for metric bitensor in place $g^{*}_{ij,kl}$.*

From Theorem [3.4] we shall take $L_{;ij;kl}$ for metric bitensor and note $g_{ij,kl}$ i. e.

$$(3.17)\qquad L_{;ij;kl}\equiv g_{ij,kl},$$

as the Finsler space, in the following theory.

Taking the intrinsic derivative of both sides of the relations $L_{;ij;kl}p^{kl}=2G_{ij}$ with $_{\alpha\beta}$, we have $L_{;ij;kl;\alpha\beta}p^{kl}=2G_{ij;\alpha\beta}-L_{;ij;kl}p^{kl}{}_{;\alpha\beta}$, from (3.1) and (3.2) the right-hand sides of the last equations vanish, consequently from (3.17) we have same relations which were obtainde by A. KAWAGUCHI[1])

$$(3.18) \qquad g_{ij,kl;\rho\tau}\, p^{kl} = g_{kl,ij;\rho\tau}\, p^{kl} = 0.$$

§ 4. The Euclidean connection.

We introduce an absolute derivation DX^i_j for a mixed tensol X^i_j being homogeneous of degree zero in p, i and j being contravariant and covariant components respectivly, in the equations

$$(4.1) \qquad DX^i_j = dX^i_j + (\Gamma_\alpha{}^i{}_h dx^h + C_\alpha{}^i{}_{kl} dp^{kl}) X^\alpha_j - (\Gamma_j{}^\beta{}_h dx^h + C_j{}^\beta{}_{kl} dp^{kl}) X^i_\beta,$$

where the two sets of functions Γ and C are subjected to such conditions as are required to ensure that they determine an Euclidean connection, when $g_{ij,kl}$ is supposed just as the metric tensor in the Finsler and Cartan spaces[2]).

At first our purposes are to determine the connection parameters Γ and C. The Euclidean connection is expressed by the condition $Dg_{ij,kl}=0$ in the sense of (4.1), from (2.6) the condition may be written in the following two equations

$$(4.2) \qquad \partial_\alpha g_{ij,kl} = \Gamma_i{}^\lambda{}_\alpha g_{\lambda j,kl} + \Gamma_j{}^\lambda{}_\alpha g_{i\lambda,kl} + \Gamma_k{}^\lambda{}_\alpha g_{ij,\lambda l} + \Gamma_l{}^\lambda{}_\alpha g_{ij,k\lambda},$$

$$(4.3) \qquad g_{ij,kl;\alpha\beta} = 2(C_i{}^\lambda{}_{\alpha\beta} g_{\lambda j,kl} + C_j{}^\lambda{}_{\alpha\beta} g_{i\lambda,kl} + C_k{}^\lambda{}_{\alpha\beta} g_{ij,\lambda l} + C_l{}^\lambda{}_{\alpha\beta} g_{ij,k\lambda}).$$

In the Finsler and Cartan spaces[3]), in order to determine the connection parameters C_{ijh}, we supposed the condition $C_{(ij)}=0$. If our space is metric class, i. e. $g_{ij,kl} = g_{ik}g_{jl} - g_{il}g_{jk}$, the condition may be written in the form $C_{(ij),\alpha\beta}=0$, then the following relations hold good

$$(4.4) \qquad C_i{}^\lambda{}_{\alpha\beta} g_{\lambda j,kl} + C_j{}^\lambda{}_{\alpha\beta} g_{i\lambda,kl} = C_k{}^\lambda{}_{\alpha\beta} g_{ij,\lambda l} + C_l{}^\lambda{}_{\alpha\beta} g_{ij,k\lambda}.$$

In this standpoint, in our general type, in order to determine $C_j{}^\lambda{}_{\alpha\beta}$ *we can suppose that the relations (4.4) are satisfied too*, then (4.3) may be written in the form

$$(4.5) \qquad g_{ij,kl;\alpha\beta} = 4(C_i{}^\lambda{}_{\alpha\beta} g_{\lambda j,kl} + C_j{}^\lambda{}_{\alpha\beta} g_{i\lambda,kl}).$$

Multiplying both sides of (4.5) by $g^{pq,kl}$ and contractiog k and l, from the relations $g_{ij,kl}g^{pq,kl} = 2\delta^{pq}_{ij}$ we have

$$(4.6) \qquad g_{ij,kl;\alpha\beta} g^{pq,kl} = 8(C_i{}^p{}_{\alpha\beta}\delta^q_j - C_i{}^q{}_{\alpha\beta}\delta^p_j + C_j{}^q{}_{\alpha\beta}\delta^p_i - C_j{}^p{}_{\alpha\beta}\delta^q_i),$$

putting $j=q$ and $i=p$ in the both sides of (4.6) and contracting same indices respectively, we have

$$(4.7) \qquad g_{ij,kl;\alpha\beta} g^{pj,kl} = 8\{(n-2)C_i{}^p{}_{\alpha\beta} + \delta^p_i C_j{}^j{}_{\alpha\beta}\},$$

$$(4.8) \qquad 16(n-1)C_i{}^i{}_{\alpha\beta} = g_{ij,kl;\alpha\beta} g^{ij,kl}.$$

From (4.7) and (4.8) we can determine the connection parameters C

1) See [12], p 77 (2.7).
2) See [1]—[3].
3) See [1]—[3].

$$C_{i\ \alpha\beta}^{\ \rho} = (n-2)^{-1}\left(\frac{1}{8} g_{ij,kl;\alpha\beta} g^{\rho j,kl} - \frac{1}{16}(n-1)^{-1}\delta_i^\rho g_{\tau j,kl;\alpha\beta} g^{\tau j,kl}\right). \tag{4.9}$$

If our space is metric class, i. e. $g^{ij,kl} = g^{ik}g^{jl} - g^{il}g^{jk}$, then (4.9) is reduced in $C_{i\ \beta\alpha}^{\ \rho} = \frac{1}{4} g_{ik;\alpha\beta}\, g^{\rho k}$, this relations are satisfied in the Finsler and Cartan spaces too[1], so we have

Theorem [4.1]. *If our space is the Finsler or Cartan space, $C_{i\ \alpha\beta}^{\ \rho}$ coincid the connection parameters of these spaces.*

Sinc the metric bitensor $g_{ij,kl}$ is homogeneous of degree zero in p, i. e. $g_{ij,kl;\alpha\beta}\, p^{\alpha\beta} = 0$, consequently we have the expectant relations which are satisfied in the Finsler and Cartan spaces, i. e.

$$C_{i\ \alpha\beta}^{\ \rho}\, p^{\alpha\beta} = 0. \tag{4.10}$$

Putting $A_{i\ \alpha\beta}^{\ \rho} \equiv F C_{i\ \alpha\beta}^{\ \rho}$, from (4.9) $C_{i\ \alpha\beta}^{\ \rho}$ is homogeneous degree -1 in p, then $A_{i\ \alpha\beta}^{\ \rho}$ is intrinsic tensor, consequently we may use the $A_{i\ \alpha\beta}^{\ \rho}$ as connection parameters for the $C_{i\ \alpha\beta}^{\ \rho}$. Multiplying both sides of (4.6) by p^{ij} and contracting i and j, from (3.18) the left-hand side of the member vanich, consequently we have

$$A_{i\ \alpha\beta}^{\ p} l^{iq} + A_{i\ \alpha\beta}^{\ q} l^{pi} = 0. \tag{4.11}$$

which were introduced by A. KAWAGUCHI and S. HOKARI[2] in the submetric class.

We shall define the unitary bivectors l^{ij} and l_{ij} in the relations

$$l_{ij} \equiv G_{ij} F^{-1}, \qquad l^{ij} \equiv p^{ij} F^{-1}, \tag{4.12}$$

then there are relations among them

$$g_{ij,kl}\, l^{ij} l^{kl} = 4, \qquad g_{ij,kl}\, l^{ij} = 2 l_{kl}, \qquad l_{ij} l^{ij} = 2. \tag{4.13}$$

From (4.10) we have $C_{i\ \alpha\beta}^{\ \rho}\, dp^{\alpha\beta} = A_{i\ \alpha\beta}^{\ \rho}\, dl^{\alpha\beta}$, and (4.1) rewrite in the equations

$$DX^i_{\ j} = dX^i_{\ j} + (\Gamma_{\alpha\ h}^{\ i} X^\alpha_{\ j} - \Gamma_{j\ h}^{\ \alpha} X^i_{\ \alpha}) dx^h + (A_{\alpha\ kl}^{\ i} X^\alpha_{\ j} - A_{j\ kl}^{\ \alpha} X^i_{\ \alpha}) dl^{kl}. \tag{4.14}$$

Substituting l^{ij} for $X^i_{\ j}$ in (4.14) and considering (4.11) we can determine the base connection, i. e.

$$Dl^{ij} = dl^{ij} + \Gamma_0{}^{ij}_{\ \ \alpha} dx^\alpha, \qquad \text{where : } \Gamma_0{}^{ij}_{\ \ \alpha} \equiv \Gamma_{\beta\ \alpha}^{\ i} l^{\beta j} + \Gamma_{\beta\ \alpha}^{\ j} l^{i\beta}. \tag{4.15}$$

Because of (4.15) equations (4.14) reduce to

$$DX^i_{\ j} = dX^i_{\ j} + (\Gamma_{\alpha\ h}^{*i} X^\alpha_{\ j} - \Gamma_{j\ h}^{*\alpha} X^i_{\ \alpha}) dx^h + (A_{\alpha\ kl}^{\ i} X^\alpha_{\ j} - A_{j\ kl}^{\ \alpha} X^i_{\ \alpha}) Dl^{kl}, \tag{4.16}$$

$$\text{where : } \Gamma_{j\ k}^{*i} \equiv \Gamma_{j\ k}^{\ i} - A_{j\ \alpha\beta}^{\ i} \Gamma_0{}^{\alpha\beta}_{\ \ k}.$$

Let us introduce a condition that $\Gamma_{j\ k}^{*i}$ are symmetric in their subscripts that is

$$\Gamma_{j\ k}^{*i} = \Gamma_{k\ j}^{*i}. \tag{4.17}$$

If both sides of (4.2) be multiplied by $g^{\rho j,kl}$ and summed for j, k and l we obtain

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1) See [1]—[3].

2) See [8].

(4. 18) $$\partial_\alpha g_{ij,kl} g^{\rho j,kl} = 2(n-2)\Gamma^{\rho}_{i\ \alpha} + 2\Gamma^{j}_{j\ \alpha}\delta^{\rho}_{i} + 2\Gamma^{\lambda}_{k\ \alpha} g_{ij,\lambda l} g^{\rho j,kl}.$$

Putting $\rho=i$ and contracting for i, we have

(4. 19) $$\Gamma^{j}_{j\ \alpha} = \frac{1}{8}(n-1)^{-1}\partial_\alpha g_{ij,kl} g^{ij,kl},$$

consequently, from (4.18) we have

(4. 20) $$\Gamma^{\rho}_{i\ \alpha} = \frac{1}{2}(n-2)^{-1}\{\partial_\alpha g_{ij,kl} g^{\rho j,kl} - \frac{1}{4}(n-1)^{-1}\partial_\alpha g_{ab,cd} g^{ab,cd}\delta^{\rho}_{i} - 2\Gamma^{\lambda}_{k\ \alpha} g_{ij,\lambda l} g^{\rho jkl}\}.$$

From (4. 17) and (4. 20) we must express the parameters Γ by fundamental functions and those derivatives, but these calculations seem to be complicated as in the submetric class[1].

§ 5. The covariant derivatives of a tensor.

We shall express the absolute derivation of X^i_j by the covariant derivatives of X^i_j. Since X^i_j is homogeneous of degree zero in p, from (2. 5) and (4. 16) if the covariant derivatives of the tensor X^i_j, in sense of the connection, are expressed by

(5. 1) $$X^i_{j|\alpha} \equiv \partial_\alpha X^i_j + \Gamma^{*i}_{\rho\ \alpha} X^\rho_j - \Gamma^{*\rho}_{j\ \alpha} X^i_\rho - \frac{1}{2} X^i_{j\cdot\lambda\mu}\Gamma^{\lambda\mu}_{0\ \ \alpha} = \partial_\alpha X^i_j + \Gamma^{*i}_{\rho\ \alpha} X^\rho_j - \Gamma^{*\rho}_{j\ \alpha} X^i_\rho - X^i_{j\cdot\lambda\mu}\Gamma^{*\lambda}_{\rho\ \alpha} l^{\rho\mu},$$

(5. 2) $$X^i_{j\|\alpha\beta} \equiv \frac{1}{2} X^i_{j\cdot\alpha\beta} + A^{\ i}_{\rho\ \alpha\beta} X^\rho_j - A^{\ \rho}_{j\ \alpha\beta} X_\rho, \qquad \text{where} : \ X^i_{j\cdot\alpha\beta} \equiv F X^i_{j;\alpha\beta},$$

then (4. 16) is rewritten in

(5. 3) $$DX^i_j = X^i_{j|\alpha}\, dx^\alpha + X^i_{j\|\alpha\beta} Dl^{\alpha\beta}.$$

Substituting $g_{ij,kl}$ in (5. 3), we have identical equations $g_{ij,kl|\alpha}\, dx^\alpha + g_{ij,kl\|\alpha\beta}\, Dl^{\alpha\beta} = 0$, since dx^α and $Dl^{\alpha\beta}$ may take mutually independently any values, those coefficients are equal to zero, that is

(5. 4) $$g_{ij,kl|\alpha} = 0, \qquad g_{ij,kl\|\alpha\beta} = 0,$$

these equations are obtained too, from (4. 2) and (4. 3).

We shall seek the covariant derivatives of unitary bivector l^{ij} and l_{ij}. From the last equations of (4. 13) we obtain the relations $l_{ij}Dl^{ij} + Dl_{ij}l^{ij} = 0$, and the second equations of (4. 13) tell us $g_{ij,kl}Dl^{kl} = 2Dl_{ij}$, cosequently we have

(5. 5) $$l_{ij}Dl^{ij} = 0, \qquad Dl_{ij}l^{ij} = 0.$$

The absolute derivations of l^{ij} are expressed by those covariant derivatives

(5. 6) $$Dl^{ij} = l^{ij}_{\ \ |\alpha} dx^\alpha + l^{ij}_{\ \ \|\alpha\beta} Dl^{\alpha\beta},$$

1) See [8], p 319, [12], p 83.

for reasons already stated in (5.4), we have

(5.7) $$l^{ij}{}_{|\alpha}=0.$$

From (4.11) and the definition (5.2) we have

(5.8) $$l^{ij}{}_{||\alpha\beta}=\frac{1}{2}l^{ij}{}_{\cdot\alpha\beta}=\frac{1}{2}F(p^{ij}F^{-1})_{;\alpha\beta}=\frac{1}{2}\{p^{ij}{}_{;\alpha\beta}+p^{ij}F(F^{-1})_{;\alpha\beta}\}$$
$$=\frac{1}{2}(p^{ij}{}_{;\alpha\beta}-p^{ij}F^{-1}F_{;\alpha\beta}),$$

(3.2) and (4.12) lead us

(5.9) $$G_{\alpha\beta}=L_{;\alpha\beta}=(F^2/2)_{;\alpha\beta}=FF_{;\alpha\beta},$$

i.e. $F_{;\alpha\beta}=l_{\alpha\beta}$, consequently, from (3.1) and (5.8) we have

(5.10) $$l^{ij}{}_{||\alpha\beta}=\frac{1}{2}l^{ij}{}_{;\alpha\beta}=2\delta^{[i}_{[\alpha}l_{\beta]\sigma}l^{j]\sigma}-l^{ij}l_{\alpha\beta}.$$

In order to express (5.7) in another form, we must adopt the following calculations ; from (4.11) and the definitions $\Gamma_0{}^{ij}{}_\alpha$ and $\Gamma^{*i}_{\rho\ \alpha}$ we have

(5.11) $$\Gamma^{*ij}_{0\ \ \alpha}=\Gamma^{\ ij}_{0\ \ \alpha} \qquad \text{where ;} \quad \Gamma^{*ij}_{0\ \ \alpha}=\Gamma^{*i}_{\rho\ \alpha}l^{\rho j}+\Gamma^{*j}_{\rho\ \alpha}l^{i\rho}.$$

From (5.1) and (5.11), (5.7) are rewitten in form $\partial_\alpha l^{ij}+\Gamma_0{}^{ij}{}_\alpha-\frac{1}{2}Fl^{ij}{}_{;\lambda\mu}\,\Gamma_0{}^{\lambda\mu}{}_\alpha=0$, because of (5.10) this relation reduces to

(5.12) $$\partial_\alpha l^{ij}=-\Gamma^{\ \lambda}_{\rho\ \alpha}l^{\rho\mu}l_{\lambda\mu}l^{ij},$$

from (4.12) we have

(5.13) $$\partial_\alpha F=F\Gamma^{\ \lambda}_{\rho\ \alpha}l^{\rho\mu}l_{\lambda\mu}.$$

The first relation of (5.4) and (5.7) lead us $l_{ij;\alpha}=0$, i.e. $\partial_\alpha l_{ij}-(\Gamma^{*\rho}_{i\ \alpha}l_{\rho j}+\Gamma^{*\rho}_{j\ \alpha}l_{i\rho})-l_{ij.\lambda\mu}\Gamma^{*\lambda}_{\rho\ \alpha}l^{\rho\mu}=0$, because of relations $l_{i\lambda\cdot j\mu}=G_{i\lambda;j\mu}-l_{i\lambda}l_{j\mu}$ we have finally

(5.14) $$\partial_\alpha l_{ij}=\Gamma^{*\rho}_{i\ \alpha}l_{\rho j}+\Gamma^{*\rho}_{j\ \alpha}l_{i\rho}+(g_{ij,\lambda\mu}-l_{ij}l_{\lambda\mu})\Gamma^{*\lambda}_{\rho\ \alpha}l^{\rho\mu}.$$

§ 6. The Curvature tensors.

We shall seek the curvature tensors by an exchange of covariant derivatives as in the usual way. In order to effect the purpose we requier many troublesome calculations; we shall show those calculations by reduced expressions.

Let X be a function being homogeneous of degree zero in p, from (2.1) and (4.12) we have $X_{;\lambda\mu}=2\dfrac{\partial X}{\partial p^{[\lambda|\rho|}}l_{\mu]\tau}l^{\rho\tau}$, consequently considering (5.13) after some calculations we obtain

(6.1) $$\partial_\beta X_{\cdot\lambda\mu}-(\partial_\beta X)_{\cdot\lambda\mu}=2F\frac{\partial X}{\partial p^{[\lambda\rho}}\partial_\beta l_{\mu]\tau}l^{\rho\tau}.$$

From (5.9) we bave $X_{\cdot\alpha\beta\cdot\gamma\delta}=l_{\gamma\delta}X_{\cdot\alpha\beta}+F^2X_{;\alpha\beta;\gamma\delta}$, because of (3.9) the last equations give us

$$(6.2)\qquad X_{\cdot\alpha\beta\cdot\gamma\delta}-X_{\cdot\gamma\delta\cdot\alpha\beta}=2X_{\cdot[\alpha|\rho|}g_{\beta]\sigma,\gamma\delta}l^{\rho\sigma}-cycl\ (\alpha\gamma),\ (\beta\delta).$$

Let X_i be a covariant vector being homogeneous of degree zero in p, since $X_{i\cdot\alpha\beta}$ is a covariant tensor, from (6.2) it is evident that

$$(6.3)\qquad X_{i\cdot\alpha\beta\|\gamma\delta}-X_{i\cdot\gamma\delta\|\alpha\beta}=X_{i\cdot[\alpha|\rho|}\,g_{\beta]\sigma,\gamma\delta}p^{\rho\sigma}-\{A_i{}^\rho{}_{\gamma\delta}X_{\rho\cdot\alpha\beta}+A_\alpha{}^\rho{}_{\gamma\delta}X_{i\cdot\rho\beta}$$
$$+A_\beta{}^\rho{}_{\gamma\delta}X_{i\cdot\alpha\rho}\}-cycl\ (\alpha\gamma),\ (\beta\delta).$$

From definition (5.2) and (6.3) we have

$$(6.4)\qquad X_{i\|\alpha\beta\|\gamma\delta}-X_{i\|\gamma\delta\|\alpha\beta}=X_\rho\,(-\frac{1}{2}A_i{}^\rho{}_{\alpha\beta\cdot\gamma\delta}-A_i{}^\sigma{}_{\alpha\beta}A_\sigma{}^\rho{}_{\gamma\delta}+A_\alpha{}^\lambda{}_{\gamma\delta}A_i{}^\rho{}_{\lambda\beta}$$
$$+A_\beta{}^\lambda{}_{\gamma\delta}A_i{}^\rho{}_{\alpha\lambda}-\frac{1}{2}(X_{i\cdot\rho\beta}A_\alpha{}^\rho{}_{\gamma\delta}+X_{i\cdot\alpha\rho}A_\beta{}^\rho{}_{\gamma\delta}-X_{i\cdot[\alpha|\rho|}\,g_{\beta]\sigma,\gamma\delta}l^{\rho\sigma})$$
$$-cycl\ (\alpha\gamma),\ (\beta\delta).$$

Differentiating both sides of (4.9) and taking the skew-symmetric parts with $(\alpha\gamma)$ and $(\beta\delta)$, and considering (3.9), (4.5) after some calculations we have

$$A_i{}^\lambda{}_{\alpha\beta;\gamma\delta}-A_i{}^\lambda{}_{\gamma\delta;\alpha\beta}=2F^{-1}A_i{}^\lambda{}_{[\alpha|\rho}g_{\beta]\sigma,\gamma\delta}\,l^{\rho\sigma}+\frac{1}{8}(n-2)^{-1}F\,g_{ia,hc;\alpha\beta}g^{\lambda a,hc}{}_{;\gamma\delta}$$
$$-cycl\ (\alpha\gamma),\ (\beta\delta).$$

Because of (4.5) and $g_{ij,\alpha\beta}\,g^{kl,\alpha\beta}=2\delta^{kl}_{ij}$ the above equations become

$$(6.5)\qquad A_i{}^\lambda{}_{\alpha\beta;\gamma\delta}-A_i{}^\lambda{}_{\gamma\delta;\alpha\beta}=2F^{-1}(A_i{}^\lambda{}_{[\alpha\rho}\,g_{\beta]\sigma,\gamma\delta}l^{\rho\sigma}-2A_i{}^\tau{}_{\alpha\beta}A_\tau{}^\lambda{}_{\gamma\delta})$$
$$-cycl\ (\alpha\gamma),\ (\beta\delta).$$

Putting (6.5) in the first term of right-hand side of (6.4) we have finally

$$(6.6)\qquad X_{i\|\alpha\beta\|\gamma\delta}-X_{i\|\gamma\delta\|\alpha\beta}=X_\rho A_i{}^\tau{}_{\alpha\beta}A_\tau{}^\rho{}_{\gamma\delta}+X_{i\cdot[\alpha|\rho|}\,g_{\beta]\sigma,\gamma\delta}l^{\rho\sigma}-2X_{i\|\rho[\beta}A_{\alpha]}{}^\rho{}_{\gamma\delta}$$
$$-cycl\ (\alpha\gamma),\ (\beta\delta).$$

We shall obtain the form of curvature tensor $R_j{}^l{}_{kl}$ by same way as in Riemannian space ; from (5.1) we have

$$X_{i|\alpha;\beta}=\partial^2_{\alpha\beta}X_i-\partial_\beta\Gamma^{*\rho}_{i\ \alpha}X_\rho-\partial_\beta X_\rho\Gamma^{*\rho}_{i\ \alpha}-\partial_\beta X_{i\cdot\lambda\mu}\Gamma^{*\lambda}_{\nu\ \alpha}l^{\nu\mu}-X_{i\cdot\lambda\mu}\partial_\beta\Gamma^{*\lambda}_{\nu\ \alpha}l^{\nu\mu}$$
$$-X_{i\cdot\lambda\mu}\Gamma^{*\lambda}_{\nu\ \alpha}\partial_\beta l^{\nu\mu}-\Gamma^{*\rho}_{i\ \beta}X_{\rho|\alpha}-\Gamma^{*\rho}_{\alpha\ \beta}X_{i|\rho}-X_{i|\alpha\cdot\lambda\mu}\Gamma^{*\lambda}_{\nu\ \beta}l^{\nu\mu},$$

considering (5.12), (5.10), (6.1) and the relations $-2l^{\alpha[i}l^{j]\beta}=l^{\alpha\beta}\,l^{ij}$, after many calculations we have

$$X_{i|\alpha|\beta}-X_{i|\beta|\alpha}=-X_\rho R_i{}^\rho{}_{\alpha\beta}-X_{i\cdot\lambda\mu}R_\nu{}^\lambda{}_{\alpha\beta}l^{\nu\mu}+\{\Gamma^{*\lambda}_{\nu\ \beta}\Gamma^{*\sigma}_{\rho\ \alpha}l^{\nu\mu}l^{\rho\tau}X_{i\cdot\sigma h,\lambda\mu}$$
$$-\Gamma^{*\lambda}_{\nu\ \alpha}l^{\nu\mu}\ (2F\frac{\partial X_i}{\partial p^{[\lambda|\rho}}\,\partial_\beta l_{\mu]\tau}l^{\rho\tau})\}-cycl\ (\alpha\beta),$$

where : $R_{i\ \beta\alpha}^{\ \rho} = \partial_\beta \Gamma^{*\rho}_{i\ \alpha} + \Gamma^{*a}_{i\ \alpha}\Gamma^{*\rho}_{a\ \beta} + \Gamma^{*\rho}_{i\ \alpha,\lambda\mu}\Gamma^{*\lambda}_{\nu\ \beta}l^{\mu\nu} - cycl\ (\alpha\beta)$,

from (5.14) after some calculations the last term { } vanishes, so we have finally

$$X_{i\ \alpha;\beta} - X_{i;\beta;\alpha} = -X_\rho R_{i\ \alpha\beta}^{\ \rho} - X_{i,\lambda\mu}R_{\nu\ \alpha\beta}^{\ \lambda}l^{\nu\mu}. \tag{6.7}$$

This result is a generalization of the *Ricci* identity in the Riemannian geometry.

Before introducing the next theories we must show

Theorem [6.1]. *Let $X(x,p)$ be a function being homogeneous degree zero in p, then there are relations*

$$dX = (\partial_\alpha X - \frac{1}{2}X_{,\rho\tau}\Gamma^{*\rho\tau}_{0\ \ \alpha})dx^\alpha + \frac{1}{2}X_{,\rho\tau}Dl^{\rho\tau}.$$

Proof : From (4.5) and (5.11) we have

$$dX = \partial_\alpha X dx^\alpha + \frac{1}{2}X_{;\rho\tau}dp^{\rho\tau} = \partial_\alpha X dx^\alpha + \frac{1}{2}X_{,\rho\tau}dl^{\rho\tau} = (\partial_\alpha X - \frac{1}{2}X_{,\rho\tau}\Gamma^{*\rho\tau}_{0\ \ \alpha})dx^\alpha$$

$$+ \frac{1}{2}X_{,\rho\tau}Dl^{\rho\tau}. \quad \text{Q. E. D.}$$

If X^i is a contravariant vector of degree zero in p, from (4.1) we have

$$\begin{aligned}(D_2D_1 - D_1D_2)X^\lambda = {} & 2\left(\frac{\partial\Gamma^{*\lambda}_{\mu\ \nu]}}{\partial x^{[\omega}} - \frac{1}{2}\Gamma^{*\lambda}_{\mu[\nu,|\rho\tau|}\Gamma^{*\rho\tau}_{0\ \ \omega} + \Gamma^{*\lambda}_{\alpha[\omega}\Gamma^{*\alpha}_{\mu|\nu]}\right)X^\mu dx_1^{\ \nu}dx_2^{\ \omega} \\ & + X^\mu(\frac{1}{2}\Gamma^{*\lambda}_{\mu\nu,\rho\tau} - \partial_\nu A_{\mu\ \rho\tau}^{\ \lambda} + \frac{1}{2}A_{\mu\ \rho\tau,ab}^{\ \lambda}\Gamma^{*ab}_{0\ \ \nu} - A_{\mu\ \rho\tau}^{\ \alpha}\Gamma^{*\lambda}_{\alpha\ \nu} + A_{\alpha\ \rho\tau}^{\ \lambda}\Gamma^{*\alpha}_{\mu\ \nu})(dx_1^{\ \nu}D_2l^{\rho\tau} \\ & - dx_2^{\ \nu}D_1l^{\rho\tau}) + X^\mu\{\frac{1}{2}A_{\mu\ \rho\tau,ab}^{\ \lambda} + A_{\mu\ \rho\tau}^{\ \alpha}A_{\alpha\ ab}^{\ \lambda} - cycl\ (\rho a),\ (\tau b)\}D_1l^{\rho\tau}D_2l^{ab} \\ & + A_{\mu\ \rho\tau}^{\ \lambda}X^\mu(d_2D_1l^{\rho\tau} - d_1D_2l^{\rho\tau}),\end{aligned} \tag{6.8}$$

where D, D_2 represent absolute derivations corresponding to two different increments d, d_2 respectively, on the other hand, for the $l^{\rho\tau}$ from (4.15) after some calculations we have

$$\begin{aligned}d_2D_1l^{\rho\tau} - d_1D_2l^{\rho\tau} = {} & 4\{(\partial_{[b}\Gamma^{*[\rho}_{k|a]} - \frac{1}{2}\Gamma^{*[\rho}_{k\ [a,cd}\Gamma^{*\ cd}_{0\ \ b]})l^{|k|\tau]} - \Gamma^{*[\rho}_{k\ [a}\Gamma^{*|k|\tau]}_{0\ \ \ b]}\} \\ & \cdot dx_2^{\ b}dx_1^{\ a} + (\Gamma^{*[\rho}_{k\ a,cd}l^{k|\tau]} + \Gamma^{*\rho}_{c\ a}\delta^\tau_d + \Gamma^{*\tau}_{d\ a}\delta^\rho_c)(D_2l^{cd}dx_1^{\ a} - D_1l^{cd}dx_2^{\ a})\end{aligned} \tag{6.9}$$

Putting (6.9) in the last term of right-hand side of (6.8) we have finally

$$\begin{aligned}2D_{[2}D_{1]}X^\lambda = {} & (R_{\mu\ ab}^{\ \lambda} + 2A_{\mu\ \rho\tau}^{\ \lambda}R_{k\ a}^{\ \rho}\ l^{k\tau})X^\mu dx_1^{\ a}dx_2^{\ b} + P_{\mu\ \nu,\rho\tau}^{\ \lambda}X^\mu(dx_1^{\ \nu}D_2l^{\rho\tau} \\ & - dx_2^{\ \nu}D_1l^{\rho\tau}) + S_{\mu\ \rho\tau,ab}^{\ \lambda}X^\mu D_1l^{\rho\tau}D_2l^{ab},\end{aligned} \tag{6.10}$$

where : $S_{\mu\ \rho\tau,ab}^{\ \lambda} \equiv -A_{\mu\ \rho\tau}^{\ \alpha}A_{\alpha\ ab}^{\ \lambda} + A_{\mu\ [\rho\alpha}^{\ \lambda}g_{\tau]\beta,ab}l^{\alpha\beta} - cycl\ (\rho a),\ (\tau b)$,

$P_{\mu\ \nu,\rho\tau}^{\ \lambda} \equiv \frac{1}{2}\Gamma^{*\lambda}_{\mu\ \nu,\rho\tau} + A_{\mu\ cd}^{\ \lambda}\Gamma^{*c}_{k\ \nu,\rho\tau}l^{kd} - A_{\mu\ \rho\tau|\nu}^{\ \lambda}$.

(September, 1953)

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