A Note on Extended Lie Systems as Riemannian Manifolds

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A Note on Extended Lie Systems as Riemannian Manifolds

Chōkō SHIBATA
Department of Mathematics, Kushiro Branch, Hokkaido University of Education
085 Kushiro

柴田鋹光：リーマン多様体としての拡大リー系における一考察

北海道教育大学釧路分校数学教室

Abstract

The purpose of the present paper is to study an extended Lie system in the strict sense which be considered as a manifold resembling very closely a simply transitive group manifold.

In § 3 we shall consider the relation between group manifolds and an extended Lie system $V_n\{\xi,a\}$ with affine Killing property and divergence property. In § 4, we give some application of a theorem of Bochner-Lichnerowicz on the Betti number of a Riemannian manifold.

§ 1. Introduction

We consider an n-uple extended Lie system $V_n\{\xi,a\}$ which is connected and paracompact. Let x^i be local coordinates in a neighbourhood of any point $p\in V$ and let (X_a) and (ω^a) be the basic frame and coframe respectively in the tangent space T_p. On an extended Lie system, we can consider the reciprocal scalars d^a_{bc} and the structure scalars a^a_{bc} [4]. An extended Lie system is called an extended Lie system in the strict sense if the following relation is satisfied:

$$(1.1)\qquad a^a_{bc}=d^a_{[bc]},$$

and we shall denote this extended Lie system by $\overset{s}{V}_n\{\zeta,a\}$ [8].

Then, we have

$$(1.2)\qquad \partial_a d^f_{bc}-\partial_c d^f_{ba}+d^e_{ba}d^f_{ec}-d^e_{bc}d^f_{ba}-a^e_{ac}d^f_{be}=0,$$

$$(1.3)\qquad a^e_{ab}a^f_{ec}+a^e_{bc}a^f_{ea}+a^e_{ca}a^f_{eb}-\partial_a a^f_{ac}-\partial_b a^f_{ca}-\partial_c a^f_{ab}=0,$$

where $a^e_{bc}=-a^e_{cb}$ and $\partial_c a^f_{ba}=\xi^i_c\dfrac{\partial a^f_{ba}}{\partial x^i}$.

§ 2. C-extended Lie system

We can put by (1.1)

$$(2.1)\qquad d^e_{ad}=a^e_{ab}+b^e_{ab}\qquad\text{where}\qquad b^e_{ab}=d^e_{(ab)}.$$

An extended Lie system in the strict sense is called a c-extended Lie system if the following relation is satisfied in (2.1),

$$b^{e}_{bc}=const.. \tag{2.2}$$

By virtue of (1.2), (2.1) and (2.2), we get

$$\begin{aligned}\partial_a a^{f}_{bc}-\partial_c a^{f}_{ba}=&\,a^{e}_{ab}a^{f}_{ec}+a^{e}_{bc}a^{f}_{ea}+a^{e}_{ca}a^{f}_{eb}+a^{e}_{ab}b^{f}_{ec}\\&+a^{e}_{bc}b^{f}_{ea}-a^{e}_{ca}b^{f}_{eb}-a^{f}_{ec}b^{e}_{ba}+a^{f}_{ea}b^{e}_{bc}-b^{e}_{ba}b^{f}_{ec}+b^{e}_{bc}b^{f}_{ea}.\end{aligned} \tag{2.3}$$

substituting (2.3) in (1.3), we have

$$\partial_b a^{e}_{ac}=b^{f}_{bc}b^{e}_{fc}+a^{f}_{bc}b^{e}_{fa}+a^{e}_{fa}b^{f}_{bc}+a^{f}_{ac}b^{e}_{bf}+a^{f}_{ab}b^{e}_{fc}+a^{e}_{cf}b^{f}_{ba}-b^{f}_{ba}b^{e}_{fc}, \tag{2.4}$$

from which and (2.3), it follows that

$$a^{f}_{ab}a^{e}_{fc}+a^{f}_{bc}a^{e}_{fa}+a^{f}_{ca}a^{e}_{fb}+a^{f}_{ab}b^{e}_{fc}+a^{f}_{bc}b^{e}_{fa}+a^{f}_{ca}b^{e}_{fb}=0. \tag{2.5}$$

Now, let δ_{ab} and ω^{a}_{b} be the components of the Riemannian metric and the connection forms with respect to the basic frames respectively. Then we have

$$d^{a}_{bc}+d^{b}_{ac}=0, \tag{2.6}$$

and if put $\omega^{a}_{b}=\Gamma^{a}_{bc}\omega^{c}$, from (1.2) and (2.1) we have

$$\Gamma^{a}_{bc}=-\frac{1}{2}d^{a}_{bc}, \tag{2.7}$$

$$R^{a}_{bcd}=\frac{1}{4}(d^{e}_{bd}d^{a}_{ec}-d^{e}_{bc}d^{a}_{ed}), \tag{2.8}$$

where R^{a}_{bcd} are the componets of the curvature tensor field with respect to the basic frame. Then, by virtue of (2.1), (2.4) and (2.5) we have

$$R^{a}_{bcd}=\frac{1}{4}(a^{a}_{bf}a^{f}_{dc}+a^{a}_{fc}b^{f}_{bd}+a^{a}_{df}b^{f}_{bc}+a^{f}_{cd}b^{a}_{fb}+b^{f}_{bd}b^{a}_{fc}-b^{f}_{bc}b^{a}_{fd}). \tag{2.9}$$

Further, from (2.6) it follows that

$$R^{a}_{bcd}=\frac{1}{4}\sum_{f}(-a^{f}_{ad}a^{f}_{cd}+a^{f}_{ad}b^{f}_{bc}-a^{f}_{ac}b^{f}_{bd}-a^{f}_{cd}b^{f}_{ad}+b^{f}_{bc}b^{f}_{ad}-b^{f}_{bd}b^{f}_{ac}). \tag{2.10}$$

Since $R_{abcd}=R_{cdab}$, we get

$$d^{e}_{ab}d^{c}_{ea}-d^{e}_{da}d^{c}_{eb}=d^{e}_{bd}d^{a}_{ec}-d^{e}_{bc}d^{a}_{ed}, \tag{2.11}$$

and

$$a^{f}_{cd}b^{f}_{ab}+a^{f}_{da}b^{f}_{bc}+a^{f}_{ac}b^{f}_{bd}=a^{f}_{ab}b^{f}_{cd}+a^{f}_{bc}b^{f}_{ad}+a^{f}_{ca}b^{f}_{bd}. \tag{2.12}$$

In a c-extended Lie system, by making use of (2.1), (2.2) and (2.6), we have

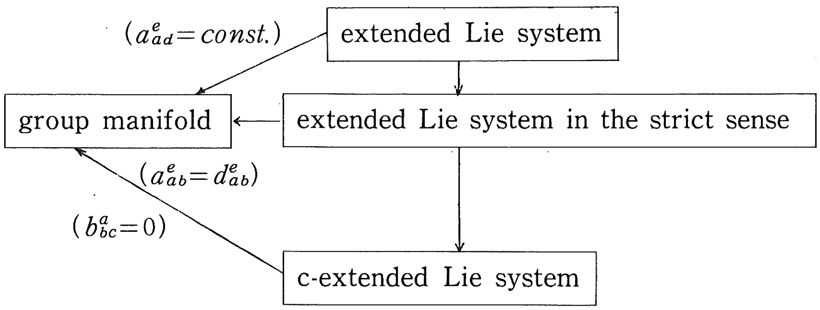
$$a^{a}_{ba}=b^{a}_{ba}=const.. \tag{2.13}$$

Here we give lemma which will be often used later.

Lemma 2.1. *An extended Lie system $V_n\{\zeta a\}$ becomes a simply transitive group manifold if and only if*

(2.14) $$b^{a}_{bc}=0.$$

Our terminologies are compared with those for extended Lie systems as follows:



§ 3. The affine Killing property and the divergence property.

We consider an n-dimensional differentiable Riemannian manifold M with metric tensor $g=(g_{ab})$. A Riemannian manifold M is said to have affine Killing property if, in some neighbour hood of each point of M, there exists an orthonomal frame $\{X_1, X_2, \cdots\cdots, X_n\}$ such that each X_a is an affine killing vector field. Such a frame is called an affine Killing frame.

Next, a necessary and sufficient condition for a vector field $V=(v^a)$ to be an affine Killing vector one is given by

$$\nabla_c\nabla_b v^a+R^{a}_{bcd}v^d=0.$$

In V_n $\{\xi, a\}$ which is, by (2.1), (2.7) and (2.8), reducible to

(3.1) $$\partial_c\partial_b v^a=\{d^{a}_{eb}\partial_c v^e+d^{a}_{ec}\partial_b v^e-d^{e}_{bc}\partial_e v^a+(\partial_c d^{a}_{fb}-b^{e}_{fb}d^{a}_{ec}+b^{a}_{fe}d^{e}_{bc})v^f\}=0.$$

For a vector field $V=C^aX_a$ (C^a ; any constant), from (3.1) which is given by

(3.2) $$(\partial_c d^{a}_{fb}-b^{e}_{bf}d^{a}_{ec}+b^{a}_{fe}d^{e}_{bc})c^f=0.$$

Here, by virtue of (2.1) we have

Proposition 3.1. *In an extended Lie system in the strict sense, a necessary and sufficient condition for a vector field C^aX_a(C^a; any constant) to be an affine Killing vector one is given by*

(3.3) $$(\partial_c d^{a}_{fb}-b^{e}_{fb}d^{a}_{ec}+b^{a}_{fe}d^{e}_{bc})c^f=0.$$

By virtue of (2.1) and (2.2), we can state

Theorem 3.2 *In a c-extended Lie system, a necessary and sufficient condition for every vector field X_a to be an affine Killing vector one is given by*

(3.4) $$\partial_c a^{a}_{eb}-b^{f}_{be}b^{a}_{fc}+b^{a}_{ef}b^{f}_{bc}-a^{a}_{fc}b^{f}_{be}+a^{f}_{bc}b^{a}_{ef}=0.$$

Next, a necessary and sufficient condition for a vector field $V=(v^a)$ to be a projective Killing vector one is that there exists a covariant vector field ψ_a such that

(3.5) $$\nabla_c \nabla_b v^a + R^a_{bcd} v^d = \psi_a \delta^a_b + \psi_c \delta^a_b$$

In this case, the components ψ_c are given by $\psi_c = \partial_c(\nabla_e v^e)/n+1$.

In[2], we know that a necessary and sufficient condition for a vector field $V = C^a X_a (C^a$; any constant) in $\overset{s}{V}\{\xi a\}$ to be a projective Killing vector one is given by

(3.6) $$C^e[\partial_c d^a_{eb} - d^f_{(be)} d^a_{fc} + d^a_{(ef)} d^f_{bc} - 2\{(\partial_b d^f_{ef})\delta^a_c + (\partial_c d^f_{ef})\delta^a_b\}/n+1] = 0$$

Then, in a c-extended Lie system , by making use of (2.1) and (2.2) we calculate (3.6) and get

(3.7) $$C^e[\partial_c a^a_{eb} - b^f_{be} b^a_{fc} + b^a_{ef} b^f_{bc} - a^a_{fc} b^f_{be} + a^f_{bc} b^a_{ef} - 2\{(\partial_b a^f_{ef})\delta^a_c + (\partial_c a^f_{ef})\delta^a_b\}/n+1] = 0.$$

Since $\partial_b a^f_{ef} = 0$ by means of (2.13), which is reducible to

(3.8) $$C^e[\partial_c a^a_{eb} - b^f_{be} b^a_{fc} + b^a_{ef} b^f_{bc} - a^a_{fc} b^f_{be} + a^f_{bc} b^a_{ef}] = 0.$$

Consequently because of Theorem 3.2 and Formula (3.8) we have

Theorem 3.3 *In a c-extended Lie system V_n $\{\xi, a\}$, a projective Killing vector field $C^a X_a (C^a$;* any constant) is an affine *Killing vector one.*

Next, an extended Lie system V_n $\{\xi, a\}$ as Riemannian manifold is said to have a divergence property if there exists a basic frame so that any one of the following equivalent conditions holds

a) $div\, X_a = 0, \quad a = 1, 2, \ldots\ldots, n$

b) $\sum_{a=1}^{n} \Gamma^a_{ba} = 0, \quad = 1, 2, \ldots\ldots, n$

Such a frame will be called divergence-free and each X_a is called an incompressible vector field[1]. A necessary and sufficient condition that $div\, X_a = 0$ for each vector field X_a in $\overset{s}{V}\{\xi, a\}$ is

(3.9) $$d^b_{ab} = 0.$$

By virtue of (2.1) and (2.2) , this condition is equivalent to

(3.10) $$a^b_{ab} + b^b_{ab} = 0.$$

Then, from (3.10) and (2.13) , we get

(3.11) $$a^b_{ab} = 0 \quad \text{and} \quad b^b_{ab} = 0.$$

Thus we can state

Proposition 3.4. *If a c-extended Lie system as Riemannian manifold has the divergence property, then it follows that*

$$a^b_{ab} = b^b_{ab} = 0.$$

Now, in (2.4) we take $e=a$ and sum on, then, by means of (3.4) and (3.11), we get

$$a^{e}_{fb}b^{f}_{ec}-b^{f}_{ce}b^{e}_{fb}=0\,. \tag{3.12}$$

Further, in (3.4) we take $e=a$ and sum on

$$a^{e}_{fb}b^{f}_{ec}+b^{f}_{ce}b^{e}_{fb}=0\,. \tag{3.13}$$

Consequently from (3.13) and (3.12), we obtain

$$b^{f}_{ce}b^{e}_{fb}=0 \quad \text{and} \quad a^{e}_{fb}b^{f}_{ec}=0\,. \tag{3.14}$$

On the other hand, for the components R_{bc} of Ricci's tensor field from (2.9) and (3.14), we have

$$R_{bc}=\frac{1}{4}(b^{e}_{fc}b^{f}_{eb}+a^{e}_{fc}a^{f}_{be}). \tag{3.15}$$

Further, by means of (3.11) and (3.14), it becomes

$$R_{bc}=-\frac{1}{4}a^{e}_{bf}a^{f}_{ce}. \tag{3.16}$$

Here if we put $\delta_{bc}=-a^{e}_{bf}a^{f}_{ce}$, then from (3.16) we have

$$R_{bc}=\frac{1}{4}\delta_{bc}\,.$$

Thus we can state

Theorem 3.5. *In a c-extended Lie system with the divergence property, if every vector field X_a is an affine Killing vector field and if we put $\delta_{bc}=-a^{e}_{bf}a^{f}_{ce}$,* then the c-extended Lie system is an *Einstein manifold.*

It is well known that in a compact and orientable Riemannian mamifold, an affine Killing vector field is a Killing vector one, and it is already shown in [4] that if evry vector field X_a is a Killing vector one, the extended Lie system in the strict sense becomes a simply transitive group manifold. Consequently from the above theorem we obtain

Corollary 3.6. *In a compact and orientable c-extended Lie system with the divergence property, if every vector field X_a is an affine Killing vector field, then the c-extended Lie system becomes a simply transitive group manifold and if we put $\delta_{bc}=-a^{e}_{bf}a^{f}_{ce}$ $(rank\,\|a^{e}_{bf}a^{f}_{ce}\|=n)$, then the c-extended Lie system is a semi-simple group manifold.*

Now, it is already show in [8] that

$$\nabla_f R^{a}_{bcd}=-\{(d^{a}_{gc}p^{g}_{fbd}-d^{a}_{gd}p^{g}_{fbc})-(d^{b}_{gc}p^{g}_{fad}-d^{a}_{gd}p^{g}_{fac})\}/8\,, \tag{3.17}$$

where

$$P^{g}_{fbd}=2\partial_f d^{g}_{bd}+d^{e}_{bf}d^{g}_{ed}+d^{e}_{df}d^{g}_{be}. \tag{3.18}$$

Let the extended Lie system in the strict sense have the affine Killing property. Then, substituting (3.4) in (3.18) we get

$$P^{g}_{fbd}=d^{e}_{db}d^{g}_{ef}-d^{e}_{df}d^{g}_{eb}+d^{e}_{bd}d^{g}_{ef}+d^{e}_{bf}d^{g}_{ed}. \tag{3.19}$$

Further, by means of (2.11), it is reducible to

$$(3.20)\qquad P^{g}_{fbd}=d^{e}_{ab}d^{f}_{eg}-d^{e}_{bg}d^{f}_{eb}+d^{e}_{bd}d^{g}_{ef}+d^{e}_{bf}d^{g}_{ed}.$$

On the other hand, from (1.1) and Bianchi's equation it becomes

$$(3.21)\qquad d^{e}_{fd}d^{g}_{eb}-d^{e}_{fb}d^{g}_{ed}+d^{e}_{ab}d^{g}_{ef}-d^{e}_{af}d^{g}_{eb}=d^{e}_{bd}d^{g}_{ef}-d^{e}_{bf}d^{g}_{ed}.$$

Thus, by virtue of (3.21) and (3.20), we have

$$(3.22)\qquad P^{g}_{fbd}=d^{e}_{fb}d^{g}_{ed}-d^{e}_{fd}d^{g}_{eb}+2d^{e}_{bd}d^{g}_{ef}.$$

Consequently substituting (3.22) in (3.17), we have

$$(3.23)\qquad \nabla_f R^{a}_{bcd}=d^{a}_{gc}(d^{e}_{fb}d^{g}_{ed}-d^{e}_{fd}d^{g}_{eb})+d^{a}_{gd}(d^{e}_{fc}d^{g}_{eb}-d^{e}_{fb}d^{g}_{ec})$$
$$+d^{b}_{gc}(d^{e}_{fd}d^{g}_{ea}-d^{e}_{fa}d^{g}_{ed})+d^{b}_{gd}(d^{e}_{fa}d^{g}_{ec}-d^{e}_{fc}d^{g}_{ea}).$$

Finally by virtue of (2.11), we can conclude

$$\nabla_f R^{a}_{bcd}=-d^{a}_{gc}d^{g}_{eg}d^{e}_{bd}+d^{b}_{gd}d^{g}_{ef}d^{e}_{ca}=0\,.$$

Here we have

Theorem 3.7. *If a c-extended Lie system* $V_n\ \{\xi,a\}$ *has the affine Killing property, then the space locally symmetric, that is, the Riemannian curvature is covarint constant.*

§4. Betti numbers

In this section, we give some applications of a theorem of Bochner-Lichnerowicz on the Betti numbers of a Riemannian manifold M. We assume that the manifold is compact and orientable. The following theorems are well known

Theorem A. *Let M be an n-dimensional compact and orientable Riemannian manifold. If the quadratic form*

$$(4.1)\qquad F_p(X)=\frac{p-1}{2}R_{abcd}X^{aba_3\cdots a_p}X^{cd}{}_{a_3\cdots a_p}+R_{bc}X^{ba_2\cdots a_p}X^{c}{}_{a_2\cdots a_p},$$

is every positive-definite, then p-th Betti number $b_p=0$ 〔7〕.

Theorem B. *(S. Tanno* 〔5〕*) Let A be a symmetric linear transformation of a vector space V, then, for every k and every $X\ \varepsilon\ V$ we have*

$$(4.2)\qquad (AX,X)^2\leqq[traceA^{2k}]^{1/k}(X,X)^2=[(A^k_{;}A^k)]^{1/k}(X,X)^2,$$

where (,) is a positive-definite inner product, and if X, Y, A etc, have indices more than one with respect to an orthonormal basis, then we understand that

$$(AX)^{ab\cdots f}=A^{a}_{c}X^{cb\cdots f},$$
$$(X,Y)=X^{ab\cdots c}Y_{ab\cdots c}\ .$$

Now, in a Riemannian manifold M, we define $T=(T_{abcd})$ thus

$$(4.3)\qquad R_{abcd}=c_1(R_{bc}\delta_{ad}-R_{bd}\delta_{ac})+c_2(R_{ad}\delta_{bc}-R_{ac}\delta_{bd})-c_3R(\delta_{bc}\delta_{ad}-\delta_{bd}\delta_{ac})+T_{abcd},$$

where $c_1, c_2, c_3 = const.$. Then, by virtue of (4.1) and (4.3), we have

$$F_P(X)=\{1-(c_1+c_2)(p-1)\}R_{bc}X^bX^c+\frac{p-1}{2}\ T_{abcd}X^{ab}X^{cd}+c_3R(p-1)X^{ab}X_{ab}.$$

If $R_{bc}\ v^b v^c$ is positive-definite, we have the smallest positive eigenvalue α of the matrix (R_{bc}). Then we have

$$F_P(X)=\{\alpha-(\alpha c_1+\alpha c_2-c_3R)(p-1)\}X^{ab}X_{ab}+\frac{p-1}{2}T_{abcd}X^{ab}X^{cd},$$

where α is an eigenvalue of the matrix (R_{bc}). Since $T_{abcd}\neq T_{adab}$ in general, we define $^*T$ by

$$(4.4)\qquad {}^*T_{abcd}=R_{abcd}-\frac{c_1+c_2}{2}(R_{bc}\delta_{ad}-R_{bd}\delta_{bc}+R_{ad}\delta_{bc}-R_{ac}\delta_{bd})+c_3R(\delta_{bc}\delta_{ad}-\delta_{bd}\delta_{ac}).$$

Then we have ${}^*T_{abcd}= {}^*T_{cdab}$ and ${}^*T_{abcd}X^{ab}X^{cd}=T_{abcd}X^{ab}X^{cd}$.

Therefore, putting $[{}^*T]=({}^*T_{(ab)(cd)}$ we have

Theorem 4.1. *Let M be a compact and orientable Riemannian manifold. If* $R_{bc}\ v^bv^c$ *is positive-definite, and for* p, $2\leqq p\leqq[n/2]$ *, it follows that*

$$(4.5)\qquad \alpha-(\alpha c_1+\alpha c_2-c_3R)(p-1)-\frac{p-1}{2}[(({}^*T]^k_,\ [{}^*T]^k)]^{1/2k}>0,$$

for some integer $k\geqq 1$*, then p-th Betti number* $b_p=0$.

From the above theorem immediately we can state

Corollary 4.2. *If* $c_1=c_2=\frac{1}{n-2}$ and $c_3=\frac{1}{(n-1)(n-2)}$ *then* T^a_{bcd} *become the components* C^a_{bcd} *of the Weyl conformal curvature tensor and (4.5) is reducible to (c.f. [5])*

$$\frac{n-2p}{n-2}\alpha+\frac{p-1}{(n-1)(n-2)}R-\frac{p-1}{2}[(([c]^k,[c]^k)]^{1/k}>0.$$

Corollary 4.3. *If* $c_1=1/n-1$*, and* $c_2=c_3=0$*, then* T^a_{bcd} *become the components* W^a_{bcd} *of the Weyl projective curvature tensor, and (4.5) is reducible to*

$$\frac{n-p}{n-1}\alpha-\frac{p-1}{2}[(([{}^*W]^k,[{}^*W]^k)]^{1/2k}>0\qquad [c,f,[5]]$$

Corollary 4.4 *If* $c_1=c_2=0$*, and* $c_3=1/n(n-1)$*, then* T^a_{bcd} *become* K^a_{bcd}*(C.f. Y. Tomonaga-[6])*

Corollary 4.5 *If* $c_1=c_2=0$, $c_3=1/n(n-1)$*, and* $_{abcd}=0$*, then M has a constant curvature.*

Next, when we consider the extended Lie system in the strict sense, by virtue of (2.10) and (4.1), we have

$$F_P(X)=R_{bc}X^bX^c+\frac{p-1}{2}R_{abcd}X^{ab}X^{cd}=R_{bc}X^bX^c+\frac{p-1}{2}(A'_{abcd}X^{ab}X^{cd}$$
$$+B'_{abcd}X^{ab}X^{cd}+C'_{abcd}X^{ab}X^{cd}),$$

where

$$(4.6)\qquad \begin{cases} A'_{abcd}=\sum_e a^e_{ab}a^e_{cd}\ , \\ B'_{abcd}=\sum_e(b^e_{bc}b^e_{ad}-b^e_{bd}b^e_{ac}), \\ C'_{abcd}=\sum_e(a^e_{cd}b^e_{ab}+a^e_{ac}b^e_{bd}+a^e_{da}b^e_{cd}). \end{cases}$$

If $R_{bc}\ v^bv^c$ is positive-definite , we have the smallest positive eigenvalue α of the matrix

(R_{bc}). Therefore we get

$$(4.7)\qquad F_P(X)\geqq \frac{1}{4}\alpha X^{a_1\cdots a_p}X_{a_1\cdots a_p}-\frac{p-1}{8}(|A'_{abcd}X^{ab}X^{cd}|+ |B'_{abcd}X^{ad}X^{cd}|+|C'_{abcd}X^{ab}X^{cd}|).$$

By means of (2.10) and (4.6), we have

$$(4.8)\qquad \begin{aligned} A'_{abcd}&=A'_{cdab},\\ B'_{abcd}&=B'_{cdab},\\ C'_{abcd}&=C'_{cdab}. \end{aligned}$$

Hence, we can apply Theorem B, and get

$$(4.9)\qquad F_P(X)\geqq \frac{1}{8}[2\alpha-(p-1)\{(traceA'^{2k})^{1/2k}+(traceB'^{2k})^{1/2k} +(traceC'^{2k})^{1/2k}]X^{a_1\cdots a_p}X_{a_1\cdots a_p}.$$

Consequently, we can state

Theorem 4.6 *Let the extended Lie system in the strict sense be compact and orientable. If $R_{bc}v^bv^c$ is positive-definite, and for p, $2\leqq p\leqq[n/2]$,*

$$2\alpha-(p-1)[(([A'_,{}^kA'^k]))^{1/2k}+(([B'_,{}^kB'^k]))^{1/2k}+(([C'_,{}^kC'^k]))^{1/2k}]>0,$$

holds for some integer $k\geqq 1$, then $b_p=0$.

If $b^e_{ab}=0$, *by* Lemma 1.2, then the extended Lie system in the strict sense is a semi-simple group manifold. In this case we have

$$-\frac{1}{4}X_{a_1\cdots a_p}X^{a_1\cdots a_p}<R_{abcd}X^{aba_3\cdots a_p}X^{cd}{}_{a_3\cdots a_p}<0,$$

and $\alpha=1/4$. Consequently from (2.10) and (4.8) it follows that

$$(4.10)\qquad 0<\frac{1}{4}A_{abcd}X^{ab}X^{cd}=R_{abcd}X^{ab}X^{cd}<X^{a_1\cdots a_p}X_{a_1\cdots a_p},$$

and $B'_{abcd}=C'_{abcd}=0$. Then the quadratic form (4.7) is, by means of (4.10), reducible to

$$F_p(X)\geqq\frac{3-p}{32}X^{a_1\cdots a_p}X_{a_1\cdots a_p}\ .$$

Thus we obtain $b_1=b_2=0$ and $b_3>0$. this being a well known fact [7].

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