A Convergence Theorem for Quasiconformal Mappings in Space

Journal of Hokkaido University of Education (Section II A) Vol. 26, No. 2, February 1976
北海道教育大学紀要（第2部 A）第26巻 第2号 昭和51年2月

A Convergence Theorem for Quasiconformal Mappings in Space

Katsuyuki OKABE

Laboratory of Mathemtics, Hakodate College, Hokkaido University of Education
040 Hakodate

岡　部　勝　幸：空間における疑等角写像に対してのある収束定理

Abstract

We generalize a convergence theorem of K. Strebel [5] for a sequence of quasiconformal mappings in the plane to a theorem for a sequence of quasiconformal mappings in n-space. If we use the three dilatations of a linear mapping, then the statement for the absolute value of the complex dilatation in the above theorem remains true in n-space. Futhermore we mention relations among the three dilatations to the convergence.

§ 1. Introduction

K. Strebel established the following theorem in the plane : *Let* (f_n) *be a sequence of K-quasiconformal mappings of a domain* D *onto a domain* D' *in the plane, which converge to a K-quasiconformal mapping* f *uniformly on every compact subset in* D. *Then* $\overline{\lim_{n\to\infty}} |K_n(z)| \geqq |K(z)|$ *a. e. in* D *where* $K(z)$ (resp. $K_n(z)$) *is the complex dilatation of* f (resp. f_n). *If equality hohs on a set* E *of positive measure, then there exists a subsequence* (n_ν) *so that* $\lim_{\nu\to\infty} K_{n_\nu}(z) = K(z)$ *a. e. on* E.

It is the purpose of this note to generalize the above theorem to a theorem with higher dimensions. The n-dimensional case can be also treated by the same method as K. Strebel's. *In particular we will borrow most of his proof.* Since we cannot consider the corresponding dilatation to the complex dilatation, the three dilatations of a linear mapping are used ([6] p. 43). The basic properties of n-dimensional quasiconformal mappings are given in [6] for $n \geqq 3$ and in [3] for $n=2$ respectively. In addition we refer to many papers and books ([1], [2], [4] and others). To many of them we are indebted.

§ 2. Preliminaries and Examples

2.1. We fist introduce some notations and terminology. Let R^n be the n-dimensional Euclidian spce. If $x \in R^n$, x_i, $i=1, 2, \dots, n$, will be the i-th coordinate of x with respect to a fixed orthogonal basis $\{e_1, e_2, \dots, e_n\}$. For a subset A of R^n we denote the closure and the complement by $\bar{A}$ and $C(A)$ respectively. If A is a measurable subset of R^n, $m_n(A) = |A|_n$ is the n-dimensional Lebsque measure. The subscript n may be omitted if there is no danger of misunderstanding. If E, F and G are subsets of R^n, the notation $\triangle(E, F, G)$ is used for the family of all paths joining E and F in G. Let Γ be a family of paths in R^n. Then Γ' denotes its image under a mapping f. Suppose that D is a domain in R^n. If a mapping $f: D \to R^n$ is differentiable at x, then a linear mapping $f'(x): R^n \to R^n$ exists at x and defined by $f'(x)e_i = \partial_i f(x)$. We use the so notations $|f'(x)| = \max_{|h|=1} |f'(x)h|$, $l(f'(x)) = \min_{|h|=1} |f'(x)h|$ and $J(x, f) = \det f'(x)$. A homeomorphism $f: D \to D'$ is said to be K-quasiconformal if and only if the following conditions are satisfied: (1) f is ACL (absolutely continuous on lines), (2) f is differentiable a. e., (3) for almost every $x \in D$ and a finite constant $K>1$, $|f'(x)|^n/K \leqq |J(x, f)| \leqq K/l(f'(x))^n$. The functions $H_1(f'(x)) = |J(x, f)|/l(f'(x))^n$, $H_0(f'(x)) = |f'(x)|^n/|J(x, f)|$ and $H(f'(x)) = |f'(x)|/l(f'(x))$, called the inner, outer and linear dilatation of $f'(x)$ respectively, are defined almost everywhere in D. We denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ positive square roots of the proper values of $f'(x)^* f'(x)$ where $f'(x)^*$ is the adjoint of $f'(x)$. We also consider mappings in the compactified n-space $\bar{R}^n = R^n \cup \{\infty\}$. A homeomorphism $f: D \to D'$ is called quasiconformal if the restriction f_1 of f to $D - \{\infty, f^{-1}(\infty)\}$ is quasiconformal. The dilatations of f are defined to be equal to the dilatations of f_1.

2. 2. Let (f_j) be a sequence of K-quasiconformal mappings of a domain D in $\bar{R}^n$, which converge locally uniformly to a mapping $f: D \to \bar{R}^n$, Then f is either a constant or a K-quasiconformal mapping onto a domain D' in $\bar{R}^n$ (Corollary 21. 3 and Corollary 37. 4 [6] or Theorem 13 [1]). In the latter case, Theorem 34. 4 and Theorem 37. 2 [6] imply that

$$\operatorname*{ess\,sup}_{x \in D} H_1(x)) \leqq \liminf_{j \to \infty} (\operatorname*{ess\,sup}_{x \in D} H_1(f_j'(x)))$$

and

$$\operatorname*{ess\,sup}_{x \in D} H_0(f'(x)) \leqq \operatorname*{iim\,inf}_{j \to \infty} (\operatorname*{ess\,sup}_{x \in D} H_0(f_j'(x))).$$

The sequences $H_1(f_j'(x))$, $H_0(f_j'(x))$ and $H(f_j'(x))$ do not always converge a. e. in D. Moreover, even if they do, it does not necessarily follow that $\lim_{j \to \infty} H_1(f_j'(x)) = H_1(f'(x))$ a.e., $\lim_{j \to \infty} H_0(f_j'(x)) = H_0(f'(x))$ a.e. and $\lim_{j \to \infty} H(f_j'(x))) = H(f'(x))$ a.e. in D. For $n=2$ O.Lehto and K. I. Virtanen gave an example so that $\lim_{h \to \infty} |K_h(z)| > |K(z)|$ a. e. ([3] p. 195). On the other hand, an example such as $\lim_{h \to \infty} |K_h(z)| < |K(z)|$ a. e. has been given by K. Strebel [5]. For $n \geqq 3$ we shall construct such exmples. They are given in R^3. For we can get them by the same way for $n \geqq 4$.

Example 1. For every finite constant $K>1$ we construct quasiconformal mappings f_n of the cube $Q = \{x | 0 < x_i < 1, i=1, 2, 3\}$ with the following properties: (1) $f_n \to x$ uniformly in Q, (2) $H_0(f_n'(x)) = K^2$ almost everywhere in Q for every n. To do this we divide Q into n^3 cubes $Q_{hkm} = \{x | (h-1)/n < x_1 < h/n, (k-1)/n < x_2 < k/n, (m-1)/n < x_3 < m/n\}$ $h, k, m = 1, 2, \ldots, n$. For $n=1, 2, \ldots$, we first define $\varphi_n(t)$ on $0 \leqq t \leqq 1/n$ by $\varphi_n(t) = Kt$ for $0 \leqq t \leqq (K+1)/n(K^2+K+1)$, $(1/K^2)t + (K^2-1)/nK^2$ for $(K+1)/n(K^2+K+1) \leqq t \leqq 1/n$ and secondarily define $\varphi_{nh}(x_1)$ on $(h-1)/n \leqq x_1 \leqq h/n$ by $\varphi_{nh}(x_1) = \varphi_n(x_1 - (h-1)/n) + (h-1)/n$ and finally define a quasiconformal mapping g_{hkm} of Q_{hkm} onto itself by $g_{hkm}(x) = (\varphi_{nk}(x_1), x_2, x_3)$. By definition, the formula $f_n(x) = g_{hkm}(x)$ for $x \in Q_{hkm}$ defines a homeomorphism f_n of Q onto itself. Since a face in R^3 has σ-finite 2-dimensional measure, by Theorem 35. 1 [6], it follows that f_n is quasiconformal with the outer dilatation $H_0(f_n'(x)) = K^2$ a.e. in Q for every n. On the other hand, since $f_n(x)$ maps Q_{hkm} onto itself, we have $|f_n(x)-x| \leqq \sqrt{3}/n$ at every point $x \in Q$. Thus we obtain a sequence f_n with the required properties.

Example 2. Let Q be the same as Example 1 and define f in Q by $f(x)=(4x_1, 2x_2, x_3)$, which has the dilatations $H_1(f'(x))=8=H_0(f'(x))$ and $H(f'(x))=4$. Moreover we define f_j in Q by $f_j(x)=(4x_1+(\sin jx_1)/j,\ 2x_2+(\sin jx_2)/j,\ x_3)$, which has the dilatations $H_1(f_j'(x))=(4+\cos jx_1)(2+\cos jx_2)$, $H_0(f_j'(x))=(4+\cos jx_1)^2/(2+\cos jx_2)$ and $H(f_j'(x))=4+\cos jx_1$. Clearly $f_j \to f$ uniformly in Q and the sequences $H_1(f_j'(x))$, $H_0(f_j'(x))$ and $H(f_j'(x))$ do not converge a.e. in Q. Since $\overline{\lim_{j\to\infty}} \cos jt=1$ a.e. and $\underline{\lim_{j\to\infty}} \cos jt=-1$ a.e. (except for every t so that t/π becomes a rational number), we have $\underline{\lim_{j\to\infty}} H_1(f_j'(x))<H(f'(x))$ a.e., $\underline{\lim_{j\to\infty}} H_0(f_j'(x))<H(f'(x))$ a.e. and $\underline{\lim_{j\to\infty}} H(f_j'(x))<H(f'(x))$ a.e. in Q.

Therefore, for the sequence $H_1(f_j'(x))$ to converge to $H_1(f'(x))$ a.e., we have no way but $\underline{\lim_{j\to\infty}} H_1(f_j'(x))=H_1(f'(x))$ a.e. on a set E. of positive measure. This is similar to the other two sequences.

3. A Convergence Theorem

We begin by giving the lemmas with no proof. We can get them by modifying Theorem 5.3, Theorem 25.1 and Theorem 25.2 [6].

Lemma 3.1 *Suppose that f is a quasiconformal mapping. Then the functions $|f'(x)|$, $l(f'(x))$, $J(x, f)$, $H_1(f'(x))$, $H_0(f'(x))$ and $H(f'(x))$ are Borel functions.*

Lemma 3.2 *Suppose that U is an open set in R and $f: U \to R$ is a quasiconformal mapping. Suppose next that $\alpha: I \to U$ is a locally rectifiable path so that f is absolutely continuous on every closed subpaths of α. Then $f\circ\alpha$ is locally rectifiable. If $e: |f\circ\alpha| \to \dot{R}^1$ is a non-negative Borel function where $|f\circ\alpha|$ is the locus of $f\circ\alpha$ and $\dot{R}^1$ is the two-point compactification $R^1 \cup \{-\infty, \infty\}$ of R^1, then*

$$\int_{f\circ\alpha} e\,ds \geqq \int_\alpha e(f(x))\, l(f'(x))\, |dx|.$$

Theorem. *Let (f_j) be a sequence of K-quasiconformal mappings of a domain D in $\bar{R}^n$ onto a domain D' in $\bar{R}^n$, which converge to a K-quasiconformal mapping f of D onto D' uniformly on every compact subset in D. Then $\overline{\lim_{j\to\infty}} H_1(f_j'(x)) \geqq H(f'(x))$ a.e. in D. If equality holds on a set E of positive measure, then there exists a subsequence (j_ν) so that $\lim_{\nu\to\infty} H_1(f_{j_\nu}'(x)) = H_1(f'(x))$ a.e. on E. This is similar to the other two sequences.*

Proof. The proof will proceed by several steps.

1. First we consider the sequence $H_0(f_j'(x))$. Let x_0 be a point in D so that f is differentiable at x_0 and the jacobian $J(x_0, f)$ of f at x_0 does not vanish. By performing a preliminary similarity transformation, we may assume that $x_0=0=f(x_0)$ and $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n > 0$. Let $Q_a(x_0)$ be a cube so that $\bar{Q}_a(x_0) \subset D$ and $Q_a(x_0)$ has the center x_0 and the length a of the edge of $Q_a(x_0)$. For each $\lambda_n/2 > \varepsilon > 0$, we can find $\delta > 0$ so that $|f(x)-f'(0)x| < a\varepsilon$ for all $a<\delta$ and $x \in Q_a(0)$. If we give a sufficiently large number j_a for every fixed $a<\delta$, then also $|f_j(x)-f'(0)x| < a\varepsilon$ for every $j > j_a$. Let A and B be the faces $x_1=-a/2$ and. $x_1=a/2$ of $Q_a(0)$ respectively and let $r_y: (-a/2, a/2) \to Q_a(0)$ be a line segment defined by $r_y(t)=y+te_1$

for $y \in G = Q_a(0) \cap \{x \in R \mid x_1 = 0\}$. Set $\Gamma = \Delta(A, B, G)$. Then $\gamma_y \in \Gamma$. Since fA lies between the hyperplanes $x_1 = (-\lambda_1/2 \pm \varepsilon)a$ and since fB lies between the hyperplanes $x_1 = (\lambda_1/2 \pm \varepsilon)a$, $l(f_0\gamma_y) \geqq (\lambda_1 - 2\varepsilon)a$ where $l(f_0\gamma_y)$ is the length of $f_0\gamma_y$ and $f_0\gamma_y \in \Gamma'$. By Theorem 4.1 and Theorem 5.3 [6], we have $a(\lambda_1 - 2\varepsilon) \leqq \int_{-a/2}^{a/2} |f_j'(y + te_1)|\, dt$. Integration over $y \in G$ yields by Fubini's theorem

$$a^n(\lambda_1 - 2\varepsilon) \leqq \int_G dm_{n-1}(y) \int_{-a/2}^{a/2} |f_j'(y + te_1)|\, dt \tag{3.1}$$

$$= \int_{Q_a} |f_j'(x)|\, dm(x).$$

Applying Hölder's inequality twice, we obtain

$$a^{n^2}(\lambda_1 - 2\varepsilon)^n \leqq \left(\int_{Q_a} |J(x, f_j)|^{\frac{1}{n-1}} dm\right)^{n-1} \left(\int_{Q_a} \frac{|f_j'(x)|^n}{|J(x, f_j)|} dm\right) \tag{3.2}$$

$$\leqq a^{n(n-2)} \int_{Q_a} |J(x, f_j)|\, dm \int_{Q_a} \frac{|f_j'(x)|^n}{|J(x, f_j)|} dm$$

$$\leqq a^{n(n-1)} (\lambda_1 + 2\varepsilon)(\lambda_2 + 2\varepsilon) \ldots\ldots (\lambda_n + 2\varepsilon) \int_{Q_a} \frac{|f_j'(x)|^n}{|J(x, f_j)|} dm.$$

Hence, for every $a < \delta$, $j > j_a$ and for almost every $x_0 \in D$, we have

$$\frac{(\lambda_1 - 2\varepsilon)^n}{(\lambda_1 + 2\varepsilon)(\lambda_2 + 2\varepsilon) \ldots (\lambda_n + 2\varepsilon)} = H_0(f'(x_0)) - o(\varepsilon) \tag{3.3}$$

$$\leqq \frac{1}{|Q_a|} \int_{Q_a(x_0)} H_0(f_j'(x))\, dm$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. Applying Fatou's lemma to the sequence of $K - H_0(f_j'(x)) \geqq 0$ in $Q_a(x_0)$, we obtain

$$\varlimsup_{j \to \infty} \int_{Q_a(x_0)} H_0(f_j'(x))\, dm \leqq \int_{Q_a(x_0)} \varlimsup_{j \to \infty} H_0(f_j'(x))\, dm. \tag{3.4}$$

(3.3) and (3.4) imply

$$H_0(f'(x_0)) - o(\varepsilon) \leqq \frac{1}{|Q_a|} \int_{Q_a(x_0)} \varlimsup_{j \to \infty} H_0(f_j'(x))\, dm. \tag{3.5}$$

Suppose that $\varlimsup_{j \to \infty} H_0(f_j'(x)) < H_0(f'(x))$ a.e. on a set E of positive measure. Then there exists a positive number d such as $\varlimsup_{j \to \infty} H_0(f_j'(x)) \leqq H_0(f'(x)) - d$ a.e. on E. Since $\varlimsup_{j \to \infty} H_0(f_j'(x)) \leqq K$ a.e. on $Q_a(x_0) - E$, we have

$$H_0(f'(x_0)) - o(\varepsilon) \leqq \frac{1}{|Q_a|} \int_{Q_a \cap E} (H_0(f'(x)) - d)\, dm + \frac{K}{|Q_a|} \int_{Q_a - E} dm$$

$$\leqq \frac{1}{|Q_a|} \int_{Q_a} (H_0(f'(x)) - d)\, dm + \frac{K}{|Q_a|} \int_{Q_a - E} dm.$$

On the other hand, the density theorem implies $\lim_{a \to 0} \frac{1}{|Q_a|} \int_{Q_a(x_0)} H_0(f'(x))\, dm = H_0(f'(x_0))$ at almost every point $x_0 \in D$. Therefore, for $a \to 0$, we first have $H_0(f'(x_0)) - o(\varepsilon) \leqq H_0(f'(x_0)) - d$ and after that, for $\varepsilon \to 0$, we have $H_0(f'(x_0)) \leqq H_0(f'(x_0)) - d$. This is a contradiction. Hence $m(E) = 0$. Namely we obtain $\varlimsup_{j \to \infty} H_0(f_j'(x)) \geqq H_0(f'(x))$ a.e. in D.

2. By Theorem 21. 11 [6], the inverse mappings f_j^{-1} converge to f^{-1} uniformly on each compact subset in D'. Then the relation $H_1(f'(x))=H_0(f^{-1\prime}(y))$, $y=f(x)$ yields $\varlimsup_{j\to\infty} H_1(f_j'(x)) \geqq H_1(f'(x))$ a. e. in D. Next we show that $\varlimsup_{j\to\infty} H(f_j'(x))\geqq H(f'(x))$ a. e. in D. From Lemma 3. 2, it follows that $a(\lambda_n+2\varepsilon)\geqq\int_{-a/2}^{a/2} l(f_j'(y+te_n))dt$ where $y\in Q_a(0)\cap\{x\in R\,|\,x_n=0\}$. By Fubini's theorem, we have

$$a^n(\lambda_n+2\varepsilon)\geqq\int_{Q_a} l(f_j'(x))dm. \tag{3.6}$$

Schwarz's inequality and (3. 6) yield

$$\frac{a^n}{\lambda_n+2\varepsilon}\leqq\int_{Q_a} l(f_j'(x))^{-1}dm. \tag{3.7}$$

By (3. 1), (3. 6), (3. 7) and Hölder's inequality, we obtain

$$\left(\frac{\lambda_1-2\varepsilon}{\lambda_n+2\varepsilon}\right)^n=H(f'(x_0))^n-o'(\varepsilon)\leqq\frac{1}{|Q_a|}\int_{Q_a(x_0)} H(f_j'(x))^n dm \tag{3.8}$$

where $o'(\varepsilon)\to 0$ and $\varepsilon\to 0$. We hold our assertion if we repeat the before-mentioned arguent for (3. 8).

3. Suppose that $\varlimsup_{j\to\infty} H_0(f_j'(x))=H_0(f'(x))$ a. e. on a set E of positive measure. If $H_0(f'(x))=1$ on E, then $1=\varliminf_{j\to\infty} H_0(f_j'(x))=\varlimsup_{j\to\infty} H_0(f_j'(x))=1$ a. e. on E. For that reason, we may assume that $H_0(f'(x))>1$ a. e. on E. By the density theorem and (3. 3), for each $\varepsilon>0$, we can find $\delta>0$ so that

$$\begin{aligned} H_0(f'(x_0))-\varepsilon &\leqq\varliminf_{j\to\infty}\frac{1}{|Q_a|}\int_{Q_a} H_0(f_j'(x))dm\leqq\varlimsup_{j\to\infty}\frac{1}{|Q_a|}\int_{Q_a} H_0(f_j'(x))dm \\ &\leqq\frac{1}{|Q_a|}\int_{Q_a}\varlimsup H_0(f_j'(x))dm=\frac{1}{|Q_a|}\int_{Q_a} H_0(f'(x))dm\leqq H_0(f'(x_0))+\varepsilon \end{aligned} \tag{3.9}$$

for every $a<\delta$ and almost every point $x_0\in E$. Set $E_r=E\cap\{x|\,|x|<r\}$. Then E_r is bounded and E_r is contained in a open set G, which satisfies $m(G-E_r)<\varepsilon$ for an arbitary $\varepsilon>0$. The closed cubes $Q_a(x)$, $x\in E_r$, $a<\delta$ cover the set E_r in the sense of Vitali. By Vitali's covering theorem, there exist a countable number of cubes $Q_{a_\nu}(x_\nu)$, $x_\nu\in E$, $a_\nu<\delta$, which are non-overlapping and satisfy $m(F-E_r)\leqq m(G-E_r)<\varepsilon$ where $E\subset F=\bigcup_\nu Q_{a_\nu}(x_\nu)\subset G$. We denot $H_1(f_j'(x))$, $H_0(f_j'(x))$ and $H(f_j'(x))$ simply by H_{1j}, H_{0j} and H_j respectively. Since (3. 9) is satisfied in each $Q_{a_\nu}(x_\nu)$, we obtain ($Q_\nu\equiv Q_{a_\nu}(x_\nu)$ and $H_{0j}(x)\equiv H_0(f_j'(x))$)

$$\begin{aligned} \sum_\nu H_0(x_\nu)a_\nu^n-\varepsilon|F| &\leqq\sum_\nu\varliminf_{j\to\infty}\int_{Q_\nu} H_{0j}\,dm\leqq\sum_\nu\varlimsup_{j\to\infty}\int_{Q_\nu} H_{0j}\,dm \\ &\leqq\sum_\nu\int_{Q_\nu}\varlimsup_{j\to\infty} H_{0j}\,dm\leqq\sum_\nu H_0(x_\nu)a_\nu^n+\varepsilon|F|. \end{aligned} \tag{3.10}$$

(3. 10) and the relations

$$\sum_\nu\varliminf_{j\to\infty}\int_{Q_\nu} H_{0j}\leqq\varliminf_{j\to\infty}\sum_\nu\int_{Q_\nu} H_{0j}=\varliminf_{j\to\infty}\int_F H_{0j}$$

and

$$\varlimsup_{j\to\infty}\int_F H_{0j}\leqq\sum_\nu\varlimsup_{j\to\infty}\int_{Q_\nu} H_{0j}$$

yield

$$(3.11)\qquad 0\leqq\int_F \overline{\lim_{j\to\infty}}\, H_{0j}-\underline{\lim_{j\to\infty}}\int_F H_{0j}<2|F|\varepsilon<2\varepsilon(|E_r|+\varepsilon).$$

On the other hand, we have

$$(3.12)\qquad 0\leqq\int_F \overline{\lim_{j\to\infty}}\, H_{0j}-\int_{E_r}\overline{\lim_{j\to\infty}}\, H_{0j}=\int_{F-E_r}\overline{\lim_{j\to\infty}}\, H_{0j}\leqq K\varepsilon$$

and

$$(3.13)\qquad 0\leqq\underline{\lim_{j\to\infty}}\int_F H_{0j}-\underline{\lim_{j\to\infty}}\int_{E_r} H_{0j}\leqq\overline{\lim_{j\to\infty}}\int_{F-E_r} H_{0j}\leqq K\varepsilon.$$

(3. 11), (3. 12) and (3. 13) imply

$$\int_{E_r} H_0-\underline{\lim_{j\to\infty}}\int_{E_r} H_{0j}=\overline{\lim_{j\to\infty}}\int_{E_r}(H_0-H_{0j})=0.$$

From $(H_0-H_{0j})^-=(H_{0j}-H_0)\cup 0$, it follows that

$$\overline{\lim_{j\to\infty}}\int_{E_r}(H_0-H_{0j})^-\leqq\int_{E_r}\overline{\lim_{j\to\infty}}\,(H_0-H_{0j})^-=0$$

Hence $\lim_{j\to\infty}\int_{E_r}|H_0-H_{0j}|\,dm=0$. This means that $H_0(f_j'(x))$ converges to $H_0(f'(x))$ in the L^1-metric. By the well-known theorem, there exists a subsequence (j_r) such as $H_{0j_r}\to H_0$ a. e. on E_r. Thus $H_0(f_{j_\nu}'(x))\to H_0(f'(x))$ a. e. on $E=\bigcup E_r$ if we take the diagonal sequence (j_ν).

4. Suppose that $\overline{\lim_{j\to\infty}}\, H_1(f_j'(x))=H_1(f'(x))$ a. e. ōn a set E of positive measure. We show that there exists a subsequence (j_ν) so that $\overline{\lim_{\nu\to\infty}}\, H_1(f_{j_\nu}'(x))=H_1(f'(x))$ a. e. on E. Choose a compact set F so that $F\subset fE$, $m(fE-F)<\varepsilon$ for an arbitary $\varepsilon>0$. By Theorem 21. 10, $f_j^{-1}|F$ are defined for large j and $f_j^{-1}\to f^{-1}$ uniformly on F. From the assumption and the relation $H_1(f'(x)=H_0(f^{-1\prime}(y))$, $y=f(x)$, it follows that $\overline{\lim_{j\to\infty}}\ H_0(f_j^{-1\prime}(y))=H_0(f^{-1\prime}(y))$ a. e. on F. Therefore there exists a subsequence (j_ν) so that $\lim_{\nu\to\infty} H_0(f_{j_\nu}^{-1\prime}(y))=H_0(f^{-1\prime}(y))$ a. e. on F. Since ε is arbitary and since a quasiconformal mapping satisfies the condition (N) (Theorem 35. 2 [6]), thus we obtain $\lim H_1(f_j'(x))=H_1(f'(x))$ a. e. on E. Next suppose that $\overline{\lim_{j\to\infty}}\, H(f_j'(x))=H(f'(x))$ a. e. on a set E. of positive measure. Repeating the before-stated argument for (3. 8), we have our assertion. Thus the proof is complete.

4. Relations among the Three Dilatations to the Convergence

The question arises whether, if one of the sequences $H_1(f_j'(x))$, $H_0(f_j'(x))$ and $H(f_j'(x))$ converges to the corresponding dilatation a. e. on a set E of positive measure, then, passing to a subsequence if necessary, the remaining two converge to the respective dilatations a. e. on E. However the following examples show that this is contradictory.

Example 3. Let $Q=\{x|0<x_i<1,\ i=1,2,3\}$ and define f and f_j by $f(x)=(4x_1, 2x_2, x_3)$ and by $f_j(x)=(4x_1, 2x_2+(\sin jx_2)/j, x_3)$ respectively. Then $f_j\to f$ uniformly in Q and $H(f_j'(x))=4=H(f'(x))$. Nevertheless the sequences $H_1(f_j'(x))$ and $H_0(f_j'(x))$ do not converge a. e. in Q.

Example 4. Let Q be the same as Example 3 and define f and f_j as follows: $f(x)=$

$(4x_1, 2x_2, x_2+x_3)$ and $f_j(x)=(4x_1, 2x_2+(\sin j(x_2+x_3))/j, x_2+x_3)$ respectively. Then $f_j \to f$ uniformly in Q. We compute the dilatations. From $\lambda_1=4$, $\lambda_2=(a^2+2a+3+\{(a^2+2a+3)^2-4\}^{1/2})^{1/2}$ and $\lambda_3=(a^2+2a+3-\{(a^2+2a+3)^2-4\}^{1/2})^{1/2}$ where $a=\cos j(x_2+x_3)$, we have $H_0(f_j'(x))=8=H_0(f'(x))$. On the other hand, the sequences $H_1(f_j'(x))$ and $H(f_j'(x))$ do not converge a. e. in Q.

If two of those sequences converge to the respective dilatations a. e. on E, then, by $H(f'(x))^n=H_1(f'(x))\,H_0(f'(x))$, it is clear that the remaining one converges to the corresponding dilatation a. e. on E. However we can state the following relation: Suppose *that* $\lim_{j\to\infty} H(f_j'(x))=H(f'(x))$ *a. e. on a set* E *of positive measure. Each sequence converges to the corresponding dilatation a. e. on* E *if one of the sequences* $H_1(f_j'(x))$ *and* $H_0(f_j'(x))$ *only converges a. e. on* E.

To prove this we assume, without loss of generality, that $\lim_{j\to\infty} H_0(f_j'(x))=H_0(f'(x))^*>H_0(f'(x))$ a. e. on E. Then $H_1(f_j'(x))$ converges a. e. on E and $\lim_{j\to\infty} H_1(f_j'(x))=H_1(f'(x))^* \geqq H_1(f'(x))$ a. e. on E. Therefore we have

$$\begin{aligned} H(f'(x))^n &= \lim_{j\to\infty} H(f_j'(x))^n \\ &= \lim_{j\to\infty} H_1(f_j'(x)) \lim_{j\to\infty} H_0(f_j'(x)) \\ &= H_1(f'(x))^* H_0(f'(x))^* > H_1(f'(x))\, H_0(f'(x)) \\ &= H(f'(x))^n, \end{aligned}$$

which is a contradiction.

References

〔1〕 Gehring, F. W. (1962), Rings and quasiconformal mappings in space. Trans. Amer. Math. Soc. Vol. 103, p. 353 – p. 393.

〔2〕 Gehring, F. W. and Väisälä, J. (1965), The coefficients of quasiconformality of domains in space. Acta Math. Vol. 114, p. 1 – 70.

〔3〕 Lehto, O und Virtanen, K. I. (1965), Quasikonformen Abbildungen Springer, Berlin. p. 269.

〔4〕 Mcshane, E. J. (1944), Integration. Princeton University press. p. 394.

〔5〕 Strebel, K. (1969), Ein Konvergenzsatz fur Folgen quasikonformer Abbildungen. Comment. Math. Helv. Vol. 44, p. 469 – p. 475.

〔6〕 Väisälä, J. (1971), Lecture on n-dimensional quasiconformal mappings. Lecture notes in math. 229, Springer, Berlin. p. 144.