



Jacobian Extensor の一般化について

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A Generalization of the Jacobian Extensor

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叶 長太郎：Jacobian Extensor の一般化について

1. **Introduction.** Prof. H. V. CRAIG has introduced the *Jacobian* extensor by the consideration of the coefficients $X_\rho^\alpha \left(X_\rho^\alpha = \binom{\alpha}{\rho} X^{(\alpha-\rho)} \right)$, in addition to the coefficients X_r^α and $X_{(\rho)r}^{(\alpha)\alpha}$ of tensor and extensor analysis. In the present paper we introduce the *generalized Jacobian extensors* and the intrinsic derivatives by the consideration of the coefficients $X_{\rho r}^{\alpha\alpha}$, in addition to the coefficients X_r^α and $X_{(\rho)r}^{(\alpha)\alpha}$.

2. **Notation.** The notations and technics employed in this paper are the same as those of Crag's paper, i. e.,

$$\begin{aligned}
 X_r^\alpha &= \frac{\partial x^\alpha}{\partial \bar{x}^r}, & X_{(\rho)r}^{(\alpha)\alpha} &= \binom{\alpha}{\rho} (X_r^\alpha)^{(\alpha-\rho)}, \\
 (2. 1) \quad X_{\rho r}^{\alpha\alpha} &= \binom{\alpha}{\rho} (\underline{X} X_r^\alpha)^{(\alpha-\rho)}, & \text{Where } a \geq \rho, \\
 &= 0 & \text{Where } a < \rho.
 \end{aligned}$$

3. **The generalized Jacobian symbols** $X_{\rho r}^{\alpha\alpha}$, $X_{\alpha\alpha}^{\rho r}$. The coefficients of the components in the transformation equation for generalized Jacobian extensors are defined as follows :

Definition 3. 1 $X_{\rho r}^{\alpha\alpha} = \binom{\alpha}{\rho} (\underline{X} X_r^\alpha)^{(\alpha-\rho)}$, $X_{\alpha\alpha}^{\rho r} = \binom{\rho}{\alpha} (\bar{X} X_\alpha^r)^{(\rho-\alpha)}$.

A very important property of those symbols is expressed by

Theorem 3. 1 *If \bar{x} , x and \tilde{x} are any three coordinate systems and if we correlate the indices λ , α and ρ to these coordinate systems respectively, then we have*

$$X_{\alpha\alpha}^{\lambda i} X_{\rho r}^{\alpha\alpha} = X_{\rho r}^{\lambda i}$$

Proof.

$$\begin{aligned}
 X_{\alpha\alpha}^{\lambda i} X_{\rho r}^{\alpha\alpha} &= \sum_{\omega=\rho}^{\lambda} \binom{\lambda}{\omega} \binom{\alpha}{\rho} (\tilde{X} X_\alpha^i)^{(\lambda-\omega)} (\underline{X} X_r^\alpha)^{(\alpha-\rho)} \\
 &= \sum_{\beta=0}^{\lambda-\rho} \binom{\lambda}{\rho} \binom{\lambda-\rho}{\beta} (\tilde{X} X_\alpha^i)^{(\beta)} (\underline{X} X_r^\alpha)^{(\lambda-\rho-\beta)} \\
 &= \binom{\lambda}{\rho} (\tilde{X} X_\alpha^i \underline{X} X_r^\alpha)^{(\lambda-\rho)} = \binom{\lambda}{\rho} (\tilde{X} X_r^i)^{(\lambda-\rho)} = X_{\rho r}^{\lambda i}
 \end{aligned}$$

for $\begin{pmatrix} \lambda \\ \rho \end{pmatrix} \begin{pmatrix} \lambda - \rho \\ \beta \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda - \beta \end{pmatrix} \begin{pmatrix} \lambda - \beta \\ \rho \end{pmatrix}$, hence the theorem is established.

An immediate consequence of this theorem is the

Corollary. $X_{\alpha\alpha}^{pr} X_{\sigma\sigma}^{\alpha\alpha} = \delta_{\sigma}^p \delta_s^r$.

4. **The generalized Jacobian extensors.** Now, we shall define the ge-ja-extensor which is the tensor relative to the extended coordinate transformation.

Definition 4. 1 Let there be given at a point P of a parameterized arc of class C^M one set of labelled numbers $E_{\beta\delta}^{\alpha\alpha, (\gamma)\epsilon, \epsilon}_f$ for each coordinate system. Further, let x and \bar{x} denote any two coordinate systems and let $E_{\beta\delta}^{\alpha\alpha, (\gamma)\epsilon, \epsilon}$ and $E_{\sigma\tau}^{pr, (\tau)\iota, u}$ be associated with systems x and \bar{x} respectively. If these quantities are related according to the equation

$$(4. 1) \quad E_{\beta\delta}^{\alpha\alpha, (\gamma)\epsilon, \epsilon} = E_{\sigma\tau}^{pr, (\tau)\iota, u} X_{\beta\beta}^{\sigma\sigma} X_{\rho\rho}^{\alpha\alpha} X_{(\gamma)\epsilon} X_{(\delta)\delta}^{(\omega)w} X_u^{\rho} X_v^{\sigma}$$

then, we shall say that are the components of an extensor of excoveriant of range M and weight W which is ge-ja-contravariant, ge-ja-covariant, excontravariant, excovariant contravariant and covariant—each of order one.

The extension of this definition to higher orders of contravariance and covariance and the generalization to the case in which not all of the Greek indices have the same range are obvious. It should be observed in passing, however, that if a ge-Jacobian superscript is confined to the value zero or a ge-Jacobian subscript to the value M , then the effect of these indices is to weight the extensor in the ordinary sense of the term. To illustrate, if a in $E_{\alpha\alpha}^{\alpha\alpha}$ is confined to the value zero, then the transformation

equation $E_{\beta\delta}^{\alpha\alpha} = E_{\rho\tau}^{pr} X_{\beta\beta}^{\alpha\alpha} X_{\rho}^{\tau}$ reduces to $E_{\beta\delta}^{\alpha\alpha} = E_{\rho\tau}^{pr} X_{\beta\beta}^{\alpha\alpha} X_{\rho}^{\tau}$,

while the introduction of the restriction $\beta = M$ into

$$E_{\rho\tau}^{\alpha\alpha} = E_{\rho\tau}^{pr} X_{\beta\beta}^{\alpha\alpha} X_u^{\rho} \text{ yields } E_{\beta\delta}^{\alpha\alpha} = E_{\rho\tau}^{pr} X_{\beta\beta}^{\alpha\alpha} X_u^{\rho}$$

Thus $E_{\beta\delta}^{\alpha\alpha}$ is a tensor of weight $-w$, while $E_{\beta\delta}^{\alpha\alpha}$ is a tensor of weight w . and we can see that the concept ge-Jacobian extensor encompasses that of weighted tensor as a special case.

Obviously, the property relating to Reduced range carry over, since it is true in general that $X_{\rho\rho}^{\alpha\alpha}$ vanishes whenever ρ exceeds α . Likes, by virtue of Transitive law, the rule relating to the sum, product and the contraction of tensors carry over. Thus, as in extensor analysis, there are $M+1$ contractions of a mixed second order ge-Jacobian extensor. We may take over without proof the following statement for the form of these contractions.

Theorem 4. 1 If $E_{\beta\delta}^{\alpha\alpha}$ is a ge-Jacobian extensor of the type indicated by the indices, then for each fixed value of θ from 0 to M , inclusive, $\sum_{\alpha=\theta}^M \binom{u}{\theta} E^{\alpha-\theta\alpha}_{\alpha\alpha}$ is an absolute scalar.

Next, as examples of ge-Jacobian extensor, we have the following

Theorem 4. 2 If $V^{\alpha\alpha}$ is a completely reduced ge-Jacobian extensor and hence of range 0 (or, in the other words, if $V^{\alpha\alpha}$ is a vector of weight $-W$), then the quantities $V^{\alpha\alpha, (\alpha)}_{,0} \leq \alpha \leq M$, constitute a ge-Jacobian extensor of the range 0 to M

Theorem 4. 3 If $V_{\alpha\alpha}$ is a completely reduced ge-Jacobian extensor of the type

indicated i , e., a vector of weight W , then the quantities $V_{\alpha a}^*$ defined by $V_{\alpha a}^* = \binom{M}{\alpha}$ $V_{M\alpha}^{(M-\alpha)}$, $0 \leq \alpha \leq M$, are the components of a ge-Jacobian extensor of range 0 to M . Since the proofs of theorem are very short and essentially the same, we give the argument for Theorem 4. 2 in outline form.

Proof. $V^{or} = V^{\alpha\alpha} X_{\alpha\alpha}^{or} = V^{\alpha\alpha} \bar{X} X_{\alpha}^r$.

Differentiating ρ times by the Leibnitz rule, we get

$$V^{or, (\rho)} = (V^{\alpha\alpha} \bar{X} X_{\alpha}^r)^{(\rho)} = \sum_{\alpha=0}^{\rho} \binom{\rho}{\alpha} V^{\alpha\alpha, (\alpha)} (\bar{X} X_{\alpha}^r)^{(\rho-\alpha)} = V^{\alpha\alpha, (\alpha)} X_{\alpha\alpha}^{pr}$$

Furthermore, we have the following statements without proof.

Theorem 4. 4 If $T^{oa, b, c}$ is a third order contravariant tensor of weight $-w$ and the necessary derivatives exist, then the quantities defined by

$$E^{\alpha\alpha, (\beta)b, (\gamma)c} = \left[\begin{matrix} \alpha. & \beta. & \gamma \\ & M & \end{matrix} \right] T^{oa, b, c, (\alpha+\beta+\gamma-2M)}$$

are the components of an extensor which is ge-ja-contravariant of order 1 and excontravariant of order 2.

Theorem 4. 5 If $T^{a, b, c, M\alpha}$ is a tensor of weight of w , then $E^{(\alpha)\alpha, (\beta)b, (\gamma)c, \delta\alpha}$ defined by

$$E^{(\alpha)\alpha, (\beta)b, (\gamma)c, \delta\alpha} = \left\{ \begin{matrix} \alpha. & \beta. & \gamma \\ M. & \delta \end{matrix} \right\} T^{a, b, c, M\alpha, (\alpha+\beta+\gamma-2M-\delta)}$$

where $\left\{ \begin{matrix} \alpha. & \beta. & \gamma \\ M. & \delta \end{matrix} \right\} = \frac{\alpha! \beta! \gamma!}{M! M! \delta! \lceil \alpha + \beta + \gamma - 2M - \delta \rceil!}$ or 0

is an extensor.

Theorem 4. 6 If $T_{M\alpha, b, c}$ is a third order covariant tensor of weight w and if $E_{\alpha\alpha, (\beta)b, (\gamma)c}$ denotes $\binom{M}{\alpha, \beta, \gamma} T_{M\alpha, b, c}^{(M-\alpha-\beta-\gamma)}$, then $E_{\alpha\alpha, (\beta)b, (\gamma)c}$ is an extensor of ge-ja-covariant order one and excovariant order two.

Thus we obtain easily the formulas from Theorems 4. 4, 4. 5 and 4. 6, by the expanding process.

$$(4. 2) \quad (T^{oa, b, c} B_b C_c)^{(\alpha)} = E^{\alpha\alpha, (\beta)b, (\gamma)c} B_{(\beta)b} C_{(\gamma)c}$$

$$(4. 3) \quad \binom{M}{\delta} (T^{a, b, c, M\alpha} A_a B_b C_c)^{(M-\delta)} = E^{(\alpha)\alpha, (\beta)b, (\gamma)c, \delta\alpha} A_{(\alpha)\alpha} B_{(\beta)b} C_{(\gamma)c}$$

$$(4. 4) \quad \binom{M}{\alpha} (T_{M\alpha, b, c} B^b C^c)^{(M-\alpha)} = E_{\alpha\alpha, (\beta)b, (\gamma)c} B^{b(\beta)} C^{c(\gamma)}$$

Finally, we note for future reference that if the extensor $A_{(\alpha)\alpha}$, $B^{b(\beta)}$, $C^{c(\gamma)}$ etc, appearing of the right members of these relationships are all expressible by extensor equations of the type $V^{\alpha(\alpha)} = V^b L^{(\alpha)a}_b$ or $V_{(\alpha)\alpha} = V_b L^b_{(\alpha)\alpha}$, then we have at once :

$$(4. 5) \quad (T^{oa, b, c} B_b C_c)^{(\alpha)} = E^{\alpha\alpha, (\beta)b, (\gamma)c} L_{(\beta)b}^f L_{(\gamma)c}^g B_f C_g$$

$$(4. 6) \quad \binom{M}{\delta} (T^{a, b, c, M\alpha} A_a B_b C_c)^{(M-\delta)} = E^{(\alpha)\alpha, (\beta)b, (\gamma)c, \delta\alpha} L_{(\alpha)\alpha}^e L_{(\beta)b}^f L_{(\gamma)c}^g A_e B_f C_g$$

$$(4. 7) \quad \binom{M}{\alpha} (T_{M\alpha, b, c} B^b C^c)^{(M-\alpha)} = E_{\alpha\alpha, (\beta)b, (\gamma)c} L^{(\beta)b}_d L^{(\gamma)c}_e B^d C^e$$

5. The ge-Jacobian connections.

At first we consider the transformation equations of the ge-Ja-extensor $J^{\rho r}_{Mb}$ and $E^{(\rho)r_s}$ ($M=1$).

$$J^{\rho r}_{Ms} = J^{\rho a}_{Mb} X^{\rho r}_{sa} X^Mb_s + J^{\rho a}_{Mb} X^{\rho r}_{sa} X^Mb_s, \quad E^{(\rho)r_s} = E^{(\rho)a_b} X^{\rho r}_{sa} X^b_s + E^{(1)a_b} X^{(\rho)r}_{sa} X^b_s.$$

In either equation if $\rho=0$, the second term drops out and the ge-Jacobian equation becomes the transformation equation for a mixed tensor while the second equation reduces to the transformation equation of a mixed tensor. When $\rho=1$, the first equation becomes

$$(5.1) \quad J^{\rho r}_{Ms} = J^{\rho a}_{Mb} \{(X^r_a)' X^b_s + X^r_a X^b_s w \cdot \log'(\bar{x} \cdot x)\} + J^{\rho a}_{Mb} X^r_a X^b_s,$$

while the second may be written

$$(5.2) \quad E^{(1)r_s} = E^{(0)a_b} (X^r_a)' X^b_s + E^{(1)a_b} X^b_s X^r_a.$$

On contracting with r and s , we obtain

$$(5.3) \quad E^{(1)r_r} = E^{(0)a_b} (X^r_a)' X^b_r + E^{(1)a_b} X^b_r.$$

Furthermore, if $E^{(0)a_b}$ and $J^{\rho a}_{Mb}$ is the Kronecker delta δ^a_b , we have

$$(5.1)' \quad J^{\rho r}_{Mb} = (X^r_a)' X^a_s + \delta^r_s W \log'(\bar{x} \cdot x) + J^{\rho a}_{Mb} X^r_a X^b_s,$$

$$(5.2)' \quad E^{(1)r_s} = (X^r_a)' X^a_s + E^{(1)a_b} X^b_s X^r_a,$$

$$(5.3)' \quad E^{(1)r_r} = (X^r_a)' X^a_r + E^{(1)a_b} X^b_r.$$

Now, according to a well known formula $\log'(\bar{x} \cdot x) = (X^r_a)' X^a_r$ and the equations (5.1)', (5.2)' and (5.3)', we assert the following statemente.

Theorem 5.1 *If $E^{(\alpha)a_b}$ is an absolute extensor of the type indicated by its indices for $M=1$ and if further, $E^{(0)a_b} = \delta^a_b$, then the quantities $J^{\alpha a}_{Mb}$ defined by $J^{\alpha a}_{Mb} = \delta^a_b$, $J^{\rho a}_{Mb} = E^{(1)a_b} + w \delta^a_b E^{(1)c}$, constitute the components of a ge-Jacobian extensor.*

The corresponding proposition for extensors of the type $E^b_{(\alpha)a}$ is as follows :

Theorem 5.2 *If $E^b_{(\alpha)a}$ is an extensor with $M=1$, $E^b_{(1)a} = \delta^b_a$ then the quantities $J^{\rho b}_{\alpha a}$ defined by $J^{\rho b}_{\alpha a} = E^b_{(\rho)a} + \delta^b_a w E^c_{(\rho)c}$, $J^{\rho b}_{\alpha a} = \delta^b_a$, are the components of a ge-Jacobian extensor.*

The proof of this theorem is similar to that of number (5.1) accordingly we omit it.

Remark. The connection extensors $L^{(\alpha)a_b}$ and $L^b_{(\alpha)a}$ defined by $L^{(0)a_b} = L^a_{(1)b} = \delta^a_b$, $L^{(1)a_b} = -L^a_b = -L^a_{bc} x^c$, $L^a_{(0)b} = L^a_b (M=1)$, are examples of extensors of the types involved in the two preceding theorems. Here the L^a_{bb} are the Christoffel symbols or, more generally, the components of a symmetric connection.

Definition 5.1 *The ge-Jacobian extensors $J^{\alpha a}_{Mb}$ and $J^{\rho b}_{\alpha a}$ which are derived from the ordinary connection extensors in accordance with the formulas :*

$$J^{\rho a}_{\alpha b} = L^a_{(\rho)b} + \delta^a_b w L^c_{(\rho)c}, \quad J^{\rho a}_{Mb} = \delta^a_b, \quad J^{\rho b}_{\alpha a} = \delta^b_a, \quad J^{\rho b}_{\alpha a} = L^{(1)b}_a + \delta^b_a w L^{(1)c}_c,$$

will be called the ge-Jacobian aconnection extensors.

As an illustration of this remark, let us consider the following proposition :

Theorem 5.3 *If S_{Ma} and S^{0a} are vectors of weights w and $-w$, respectively, and if $S^*_{\alpha a} = \binom{M}{\alpha} S_{Ma}^{(M-\alpha)}$, $S^{0a(\alpha)} = S^{\alpha a}$, then the contractions $J^{\rho a}_{\alpha b} S^{\alpha b}$ and $J^{\rho a}_{Mb} S^*_{\alpha a}$, $M=1$,*

yield the intrinsic derivatives of S_{Ma} and S^{ja} .

$$\begin{aligned} \text{Proof. } J_{Mb}^{\alpha\alpha} S_{\alpha\alpha}^* &= J_{Mb}^{0\alpha} S_{0\alpha}^* + J_{Mb}^{1\alpha} S_{1\alpha}^* = \hat{\iota}_b^\alpha S_{0\alpha}^* + (L^{(1)\alpha}_b S_{Ma} + \hat{\iota}_b^\alpha w L^{(1)\alpha}_c) S_{Ma}^* \\ &= (S_{Mb})' + L^{(1)\alpha}_b S_{Ma} + w L^{(1)\alpha}_c S_{Mb} = (S_{Mb})' - L_b^\alpha S_{Ma} - w L S_{Mb} = \frac{\partial S_{Mb}}{\partial t} \\ J_{ab}^{0\alpha} S^{\alpha b} &= J_{ab}^{0\alpha} S^{0b} + J_{ab}^{1\alpha} S^{1b} = (L_{(0)b}^\alpha + \hat{\iota}_b^\alpha w L_{(0)c}^\alpha) S_b^0 + \hat{\iota}_b^\alpha (S^{0b})' \\ &= (S^{0b})' + L_b^\alpha S^{0b} + w L S^{0b} = \frac{\partial S^{0b}}{\partial t} \end{aligned}$$

6. Extensive differentiation and the extended ge-Jacobian connection.

A moment's consideration will show that the processes upper and lower extensive differentiation, which were defined by H. V. CRAIG, have their counterparts in the present theory. Thus if $E_{\delta a}^{0\alpha}$ is a ge-Jacobian extensor, $S^{0\alpha}$ is a vector of weight $-w$, while $S^{\delta\alpha}$ denote $S^{(0)\alpha;\delta}$, then $E_{\delta a}^{0\alpha} S^{\delta\alpha}$ and its intrinsic derivative $I(E_{\delta a}^{0\alpha} S^{\delta\alpha})$ are tensors of weight $-w$ (I indicates intrinsic differentiation). Consequently, by the quotient law the multipliers of $S^{\delta\alpha}$ in $I(E_{\delta a}^{0\alpha} S^{\delta\alpha})$ constitute the components of a ge-Jacobian extensor of range $M+1$. We denote this derived extensor by the symbol $DE|_{\delta a}^{0\alpha}$ and call it the upper extensive derivative of $E_{\delta a}^{0\alpha}$.

The details of this procedure are as follows :

$$\begin{aligned} I(E_{\delta a}^{0\alpha} S^{\delta\alpha}) &= (E_{\delta a}^{0\alpha} S^{\delta\alpha})' + (L_b^\alpha + \hat{\iota}_b^\alpha w L) E_{\delta a}^{0\alpha} S^{\delta\alpha} \\ &= \sum_{\delta=0}^M (E_{\delta-1, a}^{0\alpha} + E_{\delta a}^{0\alpha}{}' + (L_b^\alpha + \hat{\iota}_b^\alpha w L) E_{\delta a}^{0\alpha}) S^{\delta\alpha} + E_{Ma}^{0\alpha} S^{M+1, a}. \end{aligned}$$

Thus we are led to formulate.

Definition 6. 1 If $E_{\delta a}^{0\alpha}$ is a ge-Jacobian extensor, then the quantities $DE|_{\delta a}^{0\alpha}$ given by the equations

$$\begin{aligned} DE|_{0a}^{0\alpha} &= E_{0a}^{0\alpha}{}' + (L_c^\alpha + \hat{\iota}_c^\alpha w L) E_{\delta a}^{0\alpha} \\ DE|_{\delta a}^{0\alpha} &= E_{\delta-1, a}^{0\alpha} + E_{\delta a}^{0\alpha}{}' + (L_c^\alpha + \hat{\iota}_c^\alpha w L) E_{\delta a}^{0\alpha} \\ DE|_{M+1, a}^{0\alpha} &= E_{Ma}^{0\alpha} \end{aligned}$$

will be called the upper extensive derivative of $E_{\delta a}^{0\alpha}$.

The lower extensive derivation may be obtained by a similar procedure. Thus we have

$$(M+1) I(E_{Mb}^{\alpha\alpha} S_{\alpha a}^*) = \sum_{\alpha=0}^{M+1} [(M-\alpha+1) E_{Mb}^{\alpha\alpha} + \alpha \{ (-L_b^\alpha - w \hat{\iota}_b^\alpha L) E_{Ma}^{\alpha-1, c} + E_{Mb}^{\alpha-1, c} \}] \bar{S}_{\alpha a},$$

where

$$\begin{aligned} (M+1) S_{\alpha a}^*{}' &= (M-\alpha+1) \bar{S}_{\alpha a}, & \bar{S}_{\alpha a} &= \binom{M+1}{\alpha} S_{Ma}^{(M-\alpha+1)}, \\ (M+1) S_{\alpha a}^* &= (\alpha+1) \bar{S}_{\alpha+1, a}, & S_{\alpha a}^* &= \binom{M}{\alpha} S_{Ma}^{(M-\alpha)}. \end{aligned}$$

The conclusion is that the quantities in the bracket constitute the components of an extensor of range zero to $M+1$.

Definition 6. 2 If $E_{Mb}^{\alpha\alpha}$ is a ge-Jacobian extensor then the quantities $D_1 E|_{M+1, b}^{\alpha\alpha}$ defined by $D_1 E|_{M+1, b}^{\alpha\alpha} = E_{Mb}^{\alpha\alpha}$; $(M+1) D_1 E|_{M+1, b}^{\alpha\alpha} = (M-\alpha+1) E_{Mb}^{\alpha\alpha} + \alpha \{ (-L_b^\alpha - \hat{\iota}_b^\alpha w L) E_{Mc}^{\alpha-1, a} + E_{Mb}^{\alpha-1, a} \}$ will be called the lower extensive derivative of the original extensor

$E_{Mb}^{\alpha\alpha}$.

Now, if we take the $M=0$ and $E_{Mb}^{0\alpha} = \delta_b^\alpha$ and apply Definition 6. 1., we get $DE|_{0b}^{0\alpha} = (L_c^\alpha + \delta_c^\alpha wL) \delta_b^\alpha = L_b^\alpha + \hat{c}_b^\alpha wL$; $DE|_{1b}^{0\alpha} = \delta_b^{\alpha*}$; the components of the ge-Jacobian connection extensor $J_{\alpha b}^{0\alpha}$. Similarly if we apply lower extensive differentiation (Definition 6. 2), we obtain the results $D_1E|_{1b}^{\alpha\alpha} = J_{1c}^{\alpha\alpha}$. As a recapitulation, we state

Theorem 6. 1 *The upper and lower extensive derivatives of \hat{c}_b^α are equal respectively to $J_{\alpha b}^{0\alpha}$ and $J_{1b}^{\alpha\alpha}$, the components of the ge-Jacobian connection extensors.*

From the development of extensive differentiation, it is obvious that if $E_{\alpha b}^{0\alpha} S^{\alpha b}$ ($S^{\alpha b} = S^{0b(\alpha)}$) gives the M -th intrinsic derivative of $S^{0\alpha}$, then $DE|_{\alpha b}^{0\alpha} S^{\alpha b}$, with a summed from zero to $M+1$, will give the intrinsic derivative of order $M+1$ of $S^{0\alpha}$. Likewise, $D_1E|_{\alpha b}^{\alpha\alpha} \bar{S}_{\alpha a}$ with a summed from 0 to $M+1$ is the intrinsic derivative of $\bar{S}_{M+1, \alpha}$ ($=S_{M\alpha}$) of order $M+1$, if $E_{Mb}^{\alpha\alpha} S_{\alpha a}$ is the M -th order intrinsic derivative of $S_{M\alpha}$. Consequently, since the first order extensive derivatives of δ_b^α constitute the components of the ge-Jacobian connection extensor, we may generate quantities E , having the properties just postulated, by repeated extensive differentiation. With this as background, we introduce.

Definition 6. 3 *The ge-Jacobian extensor generated by applying repeatedly upper and lower extensive differentiation to δ_b^α , will be denoted by the symbols $J_{\alpha b}^{0\alpha}$ and $J_{Mb}^{\alpha\alpha}$, respectively, and will be referred to as the extended ge-Jacobian connections, and assert*

Theorem 6. 2 *If $S^{0\alpha}$ is a vector of weight $-w$ and $S^{\alpha\alpha}$ denote $S^{0\alpha(\alpha)}$, then the M -th order intrinsic derivative of $S^{0\alpha}$ is given by the relationship $I^M S^{0\alpha} = J_{\alpha b}^{0\alpha} S^{\alpha b}$; and*

Theorem. 6. 3 *If $S_{M\alpha}$ is a vector of weight W and $\bar{S}_{\alpha a}$ denote $\binom{M}{\alpha} S_{M\alpha}^{(M-\alpha)}$, then the contraction $J_{M\alpha}^{\alpha b} \bar{S}_{\alpha b}$ yields the M -th order intrinsic derivative of $S_{M\alpha}$, $I^M S_{M\alpha}$.*

Thus we have the following theorem.

Theorem 6. 4 *If $T^{0\alpha, b, c}$ is a tensor of weight $-w$ and if $E^{\alpha\alpha, (\beta)b, (\gamma)c}$ is the associated extensor (given in Theorem 4. 4), then $J_{\alpha\alpha}^{0c} E^{\alpha\alpha, (\beta)b, (\gamma)c} L_{(\beta)b}^f L_{(\gamma)c}^g$ is the M -th order intrinsic derivative of $T^{0c, f, g}$.*

Proof. Let S^{0c} denote $T^{0c, b, c} B_b C_c$, with B and C arbitrary, equipollent, absolute vectors. By way of Theorem 5. 3 and equation 4. 4, we may write

$$I^M S^{0c} = J_{\alpha\alpha}^{0c} S^{0\alpha(\alpha)} = J_{\alpha\alpha}^{0c} E^{\alpha\alpha, (\beta)b, (\gamma)c} L_{(\beta)b}^f L_{(\gamma)c}^g B_f C_g,$$

on the other hand, we have

$$I^M S^{0c} = \sum \binom{M}{\beta\gamma} (I^\beta B_b) (I^\gamma C_c) I^{M-\beta-\gamma} T^{0c, b, c} = B_f C_g I^M T^{0c, f, g},$$

hence the Theorem follows by way of the arbitrariness of B and C . and similarly

Theorem 6. 5 *If $T^{e, f, g}_{M\alpha}$ is a tensor of weight w and if $E^{(\alpha)a, (\beta)b, (\gamma)c}_{\delta a}$ is the corresponding derived extensor, then $J_{M\alpha}^{\delta a} E^{(\alpha)a, (\beta)b, (\gamma)c}_{\delta a} L_{(\alpha)a}^e L_{(\beta)b}^f L_{(\gamma)c}^g$ is the M -th*

order intrinsic derivative of $T^{e, f, g}_{Ma}$.

Theorem 6.6 *If $T_{Ma, b, c}$ is a tensor of weight w and $E_{\alpha a, (\beta) b, (\gamma) c}$ is the extensor of Theorem 4.6., then*

$$I^x T_{Ma, b, c} = J^{\alpha a}_{M\beta} E_{\alpha a, (\beta) b, (\gamma) c} L^{(\beta) e}_b L^{(\gamma) f}_c.$$

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