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## A Generalization of a Theorem of Radó-Stout to Analytic Functions on Open Riemann Surfaces

Masayuki OSADA

Mathematics Laboratory, Sapporo College, Hokkaido University of Education  
Sapporo 064

長田正幸：双曲型 Riemann 面上の解析函数への Radó-Stout の定理の一般化

北海道教育大学札幌分校数学教室

### Abstract

The purpose of this paper is to establish the following theorem which is a generalization of Radó's theorem.

**THEOREM.** *Let  $E$  be a set of zero capacity in the complex plane, and  $E_o$  be a closed subset of a hyperbolic Riemann surface  $R$ . If  $f \in H_p(R - E_o)$  ( $0 < p \leq \infty$ ), i.e.  $f$  is of class  $H_p$  (Hardy class) on each component of  $R - E_o$ , if  $f$  is nonconstant on some component of  $R - E_o$ , and if for every sequence  $\{a_n\}_{n=1}^{\infty}$  in  $R - E_o$  such that  $a_n \rightarrow a_o \in E_o$  and  $\lim_{n \rightarrow \infty} f(a_n) = b_o$  exists it is the case that  $b_o \in E$ , then  $f$  has an analytic continuation to  $R$ .*

### Introduction

The following very interesting theorem of T. Radó ([5]) was proved by many authors (H. Behnke-K. Stein, H. Cartan, E. Heinz, R. Kaufmann, I. Glicksberg and etc.).

**THEOREM 1** (Theorem of Radó). *Let  $f(z)$  be a nonconstant complex-valued continuous function defined in an open unit disk  $U = \{|z| < 1\}$ . If  $f(z)$  is analytic in each component of  $U - f^{-1}(0)$ , then  $f(z)$  is analytic in  $U$ .*

E. L. Stout extended the theorem of Radó as follows.

**THEOREM 2** (Theorem of Stout). *Let  $E$  be a set of zero capacity in the complex plane, and let  $E_o$  be a relatively closed set in  $U$ . If  $f$  is a function bounded and analytic in  $U - E_o$ , if  $f'$  does not vanish identically, and if for every sequence  $\{z_n\}_{n=1}^{\infty}$  in  $U - E_o$  such that  $z_n \rightarrow z_o \in E_o$  and  $\lim_{n \rightarrow \infty} f(z_n) = w_o$  exists it is the case that  $w_o \in E$ , then  $f$  has an analytic continuation to  $U$ .*

Myron Goldstein and T. R. Chow ([3]) generalized this theorem by replacing  $U$  by an arbitrary hyperbolic Riemann surface and replacing the assumption that  $f$  is bounded by the assumption that  $f$  belongs to the Hardy class  $H_p$  ( $0 < p \leq \infty$ ) i.e.  $|f|^p$  possesses a harmonic majorant. Thus their theorem reads as follows.

THEOREM 3 (Theorem of Myron Goldstein and T. R. Chow). *Let  $E$  be a set of zero capacity in the complex plane, and let  $E_o$  be a closed subset of an open Riemann surface  $R$ . If  $f \in H_p(R - E_o)$  ( $0 < p \leq \infty$ ), i.e.  $f$  is of class  $H_p$  on each component of  $R - E_o$ , and if for every sequence  $\{a_n\}_{n=1}^\infty$  in  $R - E_o$  such that  $a_n \rightarrow a_o \in E_o$  and  $\lim_{n \rightarrow \infty} f(a_n) = b_o$  exists it is the case that  $b_o \in E$ , then  $f$  has an analytic continuation to  $R$ .*

Although their argument is very interesting and contains wide generality, it is somewhat complicated and difficult to follow. So we give here an alternating proof by a potential theoretic method.

### Preliminaries

Let  $R$  be a hyperbolic Riemann surface,  $R_M^*$  be the Martin compactification of  $R$ , and  $\Delta_1$  be the set of all minimal Martin boundary points of  $R$ . For each  $b \in \Delta_1$ , we set

$$\mathcal{G}_b = \{G \mid G \text{ is open in } R \text{ and } R - G \text{ is thin at } b\}.$$

Let  $\phi$  be a continuous mapping of  $R$  into a compact topological space  $X$ . For each  $b \in \Delta_1$ , we define

$$\phi^\wedge(b) = \bigcap_{G \in \mathcal{G}_b} \overline{\phi(G)}$$

where the closure is taken in  $X$ . If  $\phi^\wedge(b)$  is a single point of  $X$ , then we replace this by  $\widehat{\phi}(b)$ . Let  $\mathcal{F}(\phi)$  be the set of all points  $b \in \Delta_1$  for which  $\phi^\wedge(b)$  consists of a single point of  $X$ . If  $X$  is metrizable, let  $\gamma$  be a path on  $R$  terminating at  $b \in \Delta_1$ . We set

$$C(\gamma, \phi, b) = \{b' \mid b' \in X \text{ and } \lim_{n \rightarrow \infty} \phi(b_n) = b' \text{ for some sequence } \{b_n\}_{n=1}^\infty \text{ on } \gamma\}.$$

By modifying the argument in the proof of Hilfssatz 14. 1 c) in [1], we can easily prove:

LEMMA. *Let  $R$  be a hyperbolic Riemann surface, and  $X$  and  $Y$  be two compact metrizable spaces. If  $\phi$  is a continuous mapping of  $R$  into  $X$  and  $\psi$  is a continuous mapping of  $R$  into  $Y$ , then for each  $b \in \Delta_1$  there exist a path  $\gamma$  on  $R$  terminating at  $b$  such that  $C(\gamma, \phi, b) = \phi^\wedge(b)$  and  $C(\gamma, \psi, b) = \psi^\wedge(b)$ .*

COROLLARY. *In the previous lemma, suppose  $b \in \mathcal{F}(\phi) \cap \mathcal{F}(\psi)$ , then there exists a path  $\gamma$  on  $R$  terminating at  $b$  such that  $\lim_{\gamma \rightarrow b} \phi(a) = \widehat{\phi}(b)$  and  $\lim_{\gamma \rightarrow b} \psi(a) = \widehat{\psi}(b)$ .*

REMARK. As a generalization of the lemma, we can easily show the following statement: *Let  $R$  be a hyperbolic Riemann surface and  $X_n$  ( $n=1, 2, \dots$ ) be a compact metrizable space. If  $\phi_n$  ( $n=1, 2, \dots$ ) be a continuous mapping of  $R$  into  $X_n$ , then for each  $b \in \Delta_1$  there exists a path  $\gamma$  on  $R$  terminating at  $b$  such that  $C(\gamma, \phi_n, b) = \widehat{\phi}_n(b)$ .*

### Proof of theorem 3

Let  $G$  be a connected component of  $R - E_o$  on which  $f$  is nonconstant. It then follows that  $G$  is hyperbolic. Let  $\pi$  denote a mapping of  $G$  into  $R_M^*$  such that  $\pi(z) = z$  for each  $z \in G$ . Since  $\pi$  is a Fatou mapping of  $G$  into  $R$  and  $f_G$ , the restriction of  $f$  to  $G$  is a Fatou mapping of  $G$  into the complex plane  $C$ , by a theorem of Fatou (Sstoz 14.1 in [1]),  $\widehat{\pi}$  and

$\widehat{f}_G$  are defined a.e. on  $\Delta_1^G$ , where  $\Delta_1^G$  is the set of all minimal Martin boundary points of  $G$  and  $f_G$  is regarded as a continuous mapping of  $G$  into the one point compactification  $\widehat{C}$  of  $C$ . Set  $E_1 = \mathcal{F}(\pi) \cap \mathcal{F}(f_G)$  and  $E_2 = \{a | a \in E_1 \text{ and } \widehat{\pi}(a) \in E_o\}$ . Suppose  $\chi_G(E_2) > 0$ , where  $\chi_G$  denotes the canonical measure of 1 defined on  $\Delta_1^G$ . Let  $a \in \mathcal{F}(\pi) \cap \mathcal{F}(f_G)$ . By the corollary to the lemma, there exists a sequence  $\{a_n\}_{n=1}^\infty$  in  $G$  such that  $\lim_{n \rightarrow \infty} a_n = a$ ,  $\lim_{n \rightarrow \infty} \pi(a_n) = \widehat{\pi}(a)$  and  $\lim_{n \rightarrow \infty} f_G(a_n) = \widehat{f}_G(a)$ . Set  $E_3 = \widehat{f}_G(E_2)$ . By a theorem of Riesz (Satz 14.1 in [1]),  $E_3$  is of positive capacity. On the other hand, since  $E_3$  is a subset of  $E$  and  $E$  is of zero capacity,  $E_3$  is of zero capacity. This is a contradiction. Hence  $\chi_G(E_2) = 0$ , so that  $\chi_G(\{a | a \in R \text{ and } \widehat{\pi}(a) \in R\}) = 0$ . By a theorem of Constantinescu and Cornea (Satz 27 and p. 80 in [2]),  $\pi$  is of type-BI. By a theorem of Heins (Theorem 21.2 in [4]),  $E_o$  has zero capacity. By a theorem of Parreau-Yamashita (Theorem 3 in [7]),  $f$  has an analytic continuation to  $R$ .

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