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On Operators Intertwined by Dense Range Operators

Kazuyoshi OKUBO

Mathehematics Laboratory, Sapporo Callege, Hokkaido University of Education,

Sapporo 064

大久保和義：稠密な値域をもつ作用素で結ばれた作用素について

北海道教育大学札幌分校数学教室

Abstract

Let A, B, C······, be bounded linear operators on a separable complex Hilbert space. Recently, E. Goya and T. Saito showed that some properties of S are inherited by T if T, S and T*, S* are intertwined by the same operator with a dense range. In this paper, we will offer a few generalizations and remarks with regard to these facts. On the other hand, when $TW = WS$ where T is hyponormal and W has a dense range, we will give the condition of S that T is normal.

1.　In this paper, an operator means a bounded linear operator on a Hilbert space H while A, B, C, ······, denote the operators themselves. Recently, E. Goya and T. Saito showed that some properties of S are inherited by T if $TW = WS$ and $T^*W = WS^*$ where W has a dense range. F. Kubo 〔3〕 introduced the notation of algebraically definite operators. A property **S** is called abgebrically definite (resp. semidefinite) and T has the property **S**, in symbol, T∈**S** if and only if $p(T) = 0$ (resp. $p(T) \geqslant 0$) for all p∈F where F be some family of polynomials of T and T*. An operator T is algebraically definite (resp. semidefinite) if T has an algebraically definite (resp. semidefinite) property.

An operator T is called hyponormal if $TT^* \leqslant T^*T$, T is paranormal if $\| Tx \|^2 \leqslant \| T^2x \| \cdot \| x \|$ for all x ∈ H, or equivalently, $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geqslant 0$ for all $\lambda > 0$, and T is dominant if range $(T - \lambda) \subset$ range $(T^* - \bar{\lambda})$ for all $\lambda \in \sigma(T)$, or equivalently, there exists a constants M_λ such that $M_\lambda (T - \lambda)^*(T - \lambda) \geqslant (T - \lambda)(T - \lambda)^*$ for all $\lambda \in$ C. Consequently, the following properties of operators are algebraically definite or semidefinite :

ⅰ) positivity

ⅱ) selfadjointness

ⅲ) unitarity

iv) normality

v) isometry

vi) hyponormality

vii) paranormality

viii) dominantness

  2. The next theorem is a generalization of the conclusion of E. Goya and T. Saito〔2. Theorem 1〕. Our proof is in the same line as the argument in 〔6. Theorem 1〕 and 〔2. Theorem 1〕 but for the sake of convenience in the subsequent discussion we will include it.

  THEOREM 1. Let T, S and W be operators. If $TW=WS$ and $T^*W=WS^*$ where S is algebrically definite (reep. semidefinite) and W has a dense range, then T has the same algebraically definite (resp. semidefinite) property as S.

  PROOF. By polar decomposition, we can put $W^*=V^*B$ where $B=(WW^*)^{1/2}$. Since W has a dense range, it is clear that V^* is isometric. As $TW=WS$ and $T^*W=WS^*$ imply $TWW^*=WSW^*=WW^*T$, we consequently have $TB=BT$. Hence $BTV=TBV=TW=WS=BVS$ and so $TV=VS$ since B is injective. By interchanging the roles of T, S and T^*, S^* we have also $T^*V=VS^*$. Therefore we obtain $V^*VS=V^*TV=SV^*V$. Since S is algebrically definite (resp. semidefinite) there exists a family of polynomials $\{P_\lambda\}_{\lambda\in\Lambda}$ of S and S^* such that $P_\lambda(S)=0$ (resp. $P_\lambda(S)\geqslant 0$) for all $\lambda\in\Lambda$. Since $T^*V=VS^*$ and V^* is isometrical, we have $T=VSV^*$ and $T^*=VS^*V^*$. By substituting these in the polynomials $P_\lambda(T)$ of T and T^*, we can obtain $P_\lambda(T)=VP_\lambda(S)V^*=0$ (resp.$\geqslant 0$). The proof is now complete.

  Remark. In [2], it was shown in special algebraically definite (resp. semidefinite) cases that is, in the case iii), iv), v) and vi) Theorem 1 is guaranteed.

  On the other hand, in [2], it is shown that if T is a paranormal contraction, U is coisometrical, and W has a dense range, and if $TW=WU$, then T is unitary ; moreover, if W is one-to-one, then U is also unitary. (cf. [2], [4]) The problem in that without the one-to-one nature of W, can we say that U is truly unitary?.The next example shows that, in general, this is not true.

  EXAMPLE. Let H be a classical Hilbert space which is a space of all sequences of *square summable*. We define the operators W and S on H by

    $W(\xi_1,\ \xi_2,\ \xi_3,\cdots\cdots)=(\eta_1,\ \eta_2,\ \eta_3,\cdots\cdots)$

        where $\eta_{2n+1}=0(n\geq 0)$

        and $\eta_{2n}=\xi_n(n\geq 1)$

   $S(\xi_1,\ \xi_2,\ \xi_3,\cdots\cdots)=(\tau_1,\ \tau_2,\ \tau_3,\cdots\cdots)$

        where $\tau_1=0,\ \tau_{2n+1}=\xi_{2n-1}(n\gg 1)$

        and $\tau_{2n}=\xi_{2n}(n>1)$

It is easy to show that S is isometrical and W is one-to-one and $SW=W$. Consequently we have $W^*=W^*S^*$ where W^* has a dense range and S^* is coisometrical and not unitary.

   3.J. G. Stampfli and B. L. Wadhwa [6 Theorem 1] showed that if $TW=WN$ where T is dominant, N is normal and W has a dense range, then T is normal. We will present another condition of S which guarantees the normality of T if $TW=WS$ where T is hyponormal

and W has a dense range.

THEOREM 2. Let T, S and W be operators. If TW=WS where T is hyponormal and S is polynomially compact and W has a dense range, then T is normal.

PROOF. Since S is polynomially compact, there exists a polynomial p such that p(S) is compact. Let the degree of p be n. Since p(S) is compact, $\sigma(p(S))$ has at most one limit point (possibly=0). By the spectral mapping theorem, it is known that $\sigma(p(S))=p(\sigma(S))$. We will show $\sigma(S)$ has at most n limit points. Assume $\sigma(S)$ has limit points $\{\mu_1, \mu_2, \mu_3, \cdots\cdots, \mu_{n+1}\}$, then for each j $(1 \ll j \ll n+1)$ there exisis the sequence $\{\mu_m^{(j)}\}$ in $\sigma(S)$ such that

$$\mu_m^{(j)} \neq \mu_j$$

and $\quad \mu_m^{(j)} \to \mu_j (m \to \infty)$

Since p is polynomials, $p(\mu_m^{(j)}) \to p(\mu_j)$ hence $p(\mu_j)$ is a limit point of $p(\sigma(S))=\sigma(p(S))$ for each j $(1 \ll j \ll n+1)$. That is, for each j $(1 \ll j \ll n+1)$ $p(\mu_j)=0$. This is a contradiction since the degree of p is n. Thanks to S. Clary ([1]) we know that if T is hyponormal and if TW=WS where W has a dense range then $\sigma(T) \subset \sigma(S)$. Therefore we can see that $\sigma(T)$ has at most finite limit points. Following J. G. Stampfli ([5]), we can see that T is normal.

Remark. In the above proof, the condition of S is sufficient with $\sigma(S)$ has only finitely many limit points.

References

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