クロピナ空間の超曲面の誘導的,内在的理論について

On Induced and Intrinsic Theories of Hypersurfaces of Kropina Spaces.

Chōkō Shibata, U.P. Singh* and Arbind Kumar Singh*

*Mathematics Laboratory, Kushiro College, Hokkaido University of Education,

Kushiro 085

*Department of Mathematics, University of Gorakhpur, INDIA

柴田　錺光：北海道教育大学釧路分校数学教室

*ゴラクプル（インド）大学数学教室

Abstract

Let $R^n = (M^n, \alpha)$ be an n-dimensional Riemannian space with a Riemannian metric $\alpha(x, dx) = (a_{ij}(x)dx^i dx^j)^{1/2}$ and let $F^n = (M^n, L)$ be a Kropina space with the fundamental function $L(x, y) = \alpha^2(x, y)/\beta(x, y)$, where $\beta(x, dx) = b_i(x)dx^i$.

The purpose of the present paper is to study the induced and intrinsic theories of hypersurface of a Kropina space.

§0. **Introduction.** The induced and intrinsic theories of the subspaces of a Finsler space have been studied by Davies ([3]) and Rund ([9]). The connection coefficients of a Kropina hypersurface can be written as the sum of Riemannian Christoffel symbols and other tensor. In this paper we compare the induced connection coefficients with intrinsic connection coefficients of a Kropina hypersurface and discuss whether they coincide or not. The notations and terminologies are refered to Matsumoto's monograph [7].

§1. **Preliminaries.** Let F^n be an n–dimensional Kropina space. Components g_{ij} of the fundamental tensor field are given by $g_{ij} = (\partial^2 L^2/\partial y^i \partial y^j)/2$, and the covariant components $y_i = g_{ij}y^j$ of the supporting element are given by $L\partial L/\partial y^j$. The angular metric tensor $h_{ij}(x,y)$ is defined as $h_{ij} = g_{ij} - l_i l_j$, $l_i = y_i/L$. The Riemannian space R^n with the metric $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ is called the *associated Riemannian space with F^n*. The Christoffel symbols of R^n are denoted by $\{_{jk}^l\}$ and this Riemannian connection is called the *associated one*. We denote by ∇_k the covariant differentiation with respect to x_k relative to the associated Riemannian connection. The fun-

damental tensor g_{ij} and the connection coefficients $F_j{}^i{}_k$ of the Cartan connection are given respectively by ([11]).

$$(1.1) \qquad g_{ij} = \tau(2a_{ij} - l_i b_j - l_j b_i) + l_i l_j, \qquad \tau = \alpha^2 \beta^{-2}$$

$$(1.2) \qquad F_j{}^i{}_k = \{_j{}^i{}_k\} + D_j{}^i{}_k.$$

The tensor D_{jk}^i, called the *difference tensor*, is given by ([11])

$$(1.3) \qquad D_j{}^i{}_k = -Q^{ir}(F_{rj}l_k + F_{rk}l_j) - E_{jk}Q^i - h^i{}_j \varPhi_k$$

$$-h^i{}_k \varPhi_j + h_{jk}\varPhi^i + \lambda C_j{}^i{}_k,$$

where we put

$$
\begin{aligned}
&(1) \quad \nabla_k b_j = b_{jk}, \quad 2E_{jk} = b_{jk} + b_{kj}, \quad 2F_{jk} = b_{jk} - b_{kj},\\
&(2) \quad b^i = a^{ij}b_j, \quad a^{ij}a_{jk} = \delta^i{}_k, \quad \rho = a_{ij}b^i b^j,\\
(1.4) \quad &\qquad Q^i = (2l^i - b^i)/\rho, \quad Q^{ir} = (a^{ir} + Q^i b^r)/2,\\
&(3) \quad \varPhi_k = (\rho F_{0k}/\beta - \rho Q^r F_{rk} + 2b_{0k}/L + F_{r0}b^r l_k/L)/2,\\
&\qquad g^{ir}\varPhi_r = \varPhi^i, \quad \lambda = (E_{00}/L + F_{r0}b^r)/\rho, \quad \varPhi_k y^k = \lambda.
\end{aligned}
$$

In (1.4) 3) and the remainder of the present paper the suffix " 0 " means the contraction by y^i. Contraction of (1.3) by y^k gives

$$(1.5) \qquad D_j{}^i{}_0 = -\{a^{ir}(LF_{rj} + F_{r0}l_j) + b^r(2l^i - b^i)(LF_{rj} +$$

$$+ F_{r0}l_j)/\rho\}/2 - E_{j0}(2l^i - b^i)/\rho - \lambda h^i{}_j,$$

where (1.4) was used.

Lemma 1([11]). *The difference tensor D_{jk}^i vanishes if and only if the covariant vector b_i is parallel with respect to the associated Riemannian connection, i.e., $\nabla_k b_i = 0$.*

§ 2. Hypersurfaces of Kropina space and associated Riemannian space.

First, we are concerned with a hypersurface H^{n-1} of the underlying manifold M^n of a Kropina space $F^n = (M^n, L)$, which is represented parametrically by

$$(2.1) \qquad x^i = x^i(u^\sigma), \qquad \sigma = 1, 2, \cdots, n-1,$$

where u^σ are Gaussian coordinates on H^{n-1}. Introducing the notations

$$(2.2) \qquad B_\alpha^i = \partial x^i / \partial u^\alpha,$$

we shall assume that the matrix of these projection factors is of rank $n-1$. The following notations are also employed:

(2.3) $B_{\alpha\beta}^i = \partial^2 x^i / \partial u^\alpha \partial u^\beta$, $B_{\alpha\beta\cdots\gamma}^{ij\cdots k} = B_\alpha^i B_\beta^j \cdots B_\gamma^k$.

The functions $B_\alpha^i(x)$ may be considered as components of $n-1$ linearly independent vectors tangent to H^{n-1}. Therefore any vector x^i, tangent to H^{n-1}, may be written uniquely in the form

(2.4) $x^i = B_\alpha^i x^\alpha$,

where X^α are components of the vector relative to the u-coordinate system. In particular, we assume that the supporting element y^i is tangential to H^{n-1} so that

(2.5) $y^i (=\dot{x}^i) = B_\alpha^i \dot{u}^\alpha$.

The induced fundamental metric tensor $g_{\alpha\beta}(u, \dot{u})$ of the hypersurface H^{n-1} defined with respect to such a direction is given by

(2.6) $g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, y) B_{\alpha\beta}^{ij}$.

If $L(x, y)$ represents the fundamental function of F^n for a direction y^i tangent to H^{n-1}, it follows from (2.5) that the corresponding fundamental function for H^{n-1} is given by $\overline{L}(u, \dot{u}) = L(x^i(u), B_\alpha^i \dot{u}^\alpha)$. For the Kropina space F^n, it follows from (2.1) and (2.5) that the fundamental function $\overline{L}$ is given by

(2.7) $\overline{L}(u, \dot{u}) = a_{\alpha\beta}(u) \dot{u}^\alpha \dot{u}^\beta / b_\beta \dot{u}^\beta$, $a_{\alpha\beta} = a_{ij} B_{\alpha\beta}^{ij}$,

in which $a_{\alpha\beta}(u)$ is the fundamental tensor of the Riemannian hypersurface R^{n-1} and $b_\alpha(u)$ is given by

(2.8) $b_\alpha = b_i B_\alpha^i$.

Thus, in virtue of (1.1), (2.7) and (2.8), the induced metric tensor $g_{\alpha\beta}$ in (2.6) is written by

(2.6′) $g_{\alpha\beta} = \overline{\tau}(2a_{\alpha\beta} - l_\alpha b_\beta - l_\beta b_\alpha) + l_\alpha l_\beta$, $\overline{\tau} = \tau$.

Here we have

Proposition 1. *A hypersurface of a Kropina space is also a Kropina space.*

Remark. From the above proposition, the hypersurface of a Kropina space is called a *Kropina hypersurface.*

Further, we have

(2.9) $l^i = B_\alpha^i l^\alpha$.

As usual, $\det(g_{ij}) \neq 0$ is supposed. Thus according to our assumption the tensor $g_{\alpha\beta}(u, \dot{u})$ possesses the reciprocal tensor $g_{\alpha\beta}$ which is used to define a set of $n-1$ covariant vectors

(2.10) $B_i^\alpha(x, y) = g^{\alpha\beta}(u, \dot{u}) g_{ij}(x, y) B_\beta^j(x)$,

which satisfy

(2.11) $B_\alpha^l B_l^\beta = \delta_\alpha^\beta.$

Another useful indentity ([3]) is

(2.12) $B_l^\alpha B_\alpha^j = \delta_l^j - N_l N^j,$

where the unit normal vector $N^l(x, y)$ is defined at each point of the Kropina hypersurface H^{n-1} with respect to the tangential supporting element y^l by a system of equations

(2.13) $N^l = g^{lj}(x, y) N_j, \quad g_{lj} N^l N^j = 1, \quad N_l B_\alpha^l = 0,$

which in turn imply

(2.14) $N^l B_l^\alpha = 0.$

Further we get

(2.15) $g_{lj} = g_{\alpha\beta} B_l^\alpha B_j^\beta + N_l N_j, \quad g^{lj} = g^{\alpha\beta} B_\alpha^l B_\beta^j + N^j N^l.$

Next, we shall consider a hypersurface R^{n-1} of the associated Riemannian space with the metric $\alpha = (a_{lj}(x) y^l y^j)^{1/2}$ represented parametrically by the same equations as (2.1). Then u^α in (2.1) are Gaussian coordinates on R^{n-1}. And the function $B_\alpha^l(x)$ in (2.2) may be considered to be components of a set of $n-1$ linearly independent vectors tangent to R^{n-1}. The induced fundamental metric tensor of the Riemannian hypersurface R^{n-1} is given by $a_{\alpha\beta}$ in (2.7). The hypersurface of the associated Riemannian space R^n is called an *associated Riemannian hypersurface* $R^{n-1} = (M^{n-1}, \quad \overline{\alpha} = (a_{\alpha\beta}(u) \dot{u}^\alpha \dot{u}^\beta)^{1/2})$.
The quantities $\overline{B}_l^\alpha(x)$ are uniquely defined along R^{n-1} by the equations

(2.16) $\overline{B}_l^\alpha(x) = a^{\alpha\beta}(u) a_{lj} B_\beta^j(x).$

We denote the covariant components of a unit normal vector of R^{n-1} by $\overline{N}^l$. Then we have a field of linear frame $(B_1^l, \cdots, B_{n-1}^l; \overline{N}^l = a^{lj} \overline{N}_j)$ of R^n defined along R^{n-1} by

(2.17) $B_\alpha^l \overline{B}_l^\beta = \delta_\alpha^\beta, \quad B_\alpha^l \overline{B}_j^\alpha = \delta_j^l - \overline{N}^l \overline{N}_j,, \quad \overline{N}^l \overline{B}_l^\alpha = 0.$

It follows from (2.17) that

(2.18) $a_{lj} = a_{\alpha\beta} \overline{B}_l^\alpha \overline{B}_j^\beta + \overline{N}_l \overline{N}_j$

Since $\overline{N}_l B_\alpha^l = 0$ and $B_\alpha^l \dot{u}^\alpha = y^l$, we see that the supporting element y^l is tangential to the associated Riemannian hypersurface R^{n-1}, that is $\overline{N}_l y^l = 0$, so that we have

(2.19) $\overline{N}^l Y_l = 0, \quad Y_l = a_{lj} y^j,$

which will play an important role later on. The reciprocal tensor $g^{\alpha\beta}$ of $g_{\alpha\beta}$ is given by

(2.20) $g^{\alpha\beta} = [\overline{\rho} a^{\alpha\beta} + 2(1^\alpha b^\beta + 1^\beta b^\alpha) - b^\alpha b^\beta + 2(\overline{\rho}\overline{\tau} - 2)l^\alpha l^\beta]/2\overline{\rho}\overline{\tau},$

where we put

(2.21)　(a)　$\bar{\rho} = a_{\alpha\beta}b^{\alpha}b^{\beta}$,　(b)　$a^{\alpha\beta}b_{\alpha} = b^{\beta}$,　(c)　$1^{\alpha} = g^{\alpha\beta}1_{\beta}$.

With the help of relations (1.4)2), (2.7), (2.8) and (2.21)a), we can easily obtain

(2.22)　$\bar{\rho} = \rho - (b_i\overline{N}^i)^2$.

It follows from (2.6'), (2.20), (2.21) and (2.22) that

(2.23)
(1)　$Y^{\alpha} = (\dot{u}^{\alpha})/\beta = \tau 1^{\alpha}$,　(2)　$1_{\alpha}b^{\alpha} = 2 - \bar{\rho}\tau$,

(3)　$a^{\alpha\epsilon}1_{\epsilon} = 2Y^{\alpha} - \tau b^{\alpha}$,　(4)　$b^j = b^{\alpha}B_{\alpha}^j + (b_i\overline{N}^i)\overline{N}^j$.

Further, in virtue of (2.10), (2.16) and (2.23) we have

(2.24)　$B_i^{\alpha} = \overline{B}_i^{\alpha} + (b_m\overline{N}^m)(b^{\alpha} - 21^{\alpha})\overline{N}_i/\bar{\rho}$.

§3. Relation between induced and intrinsic connection parameters.

The Cartan connection coefficients of the Finsler space F^n are denoted by $F_j{}^i{}_k$. The induced connection parameters of hypersurface are defined by the relation ([8])

(3.1)　$F_{\gamma}{}^{\alpha}{}_{\beta} = B_i^{\alpha}(B_{\beta\gamma}^i + F_j{}^i{}_k B_{\beta\gamma}^{jk})$.

And the intrinsic connection coefficients $\overline{F}_{\beta}{}^{\alpha}{}_{\gamma}$ are defined with respect to the induced metric (2.6) of hypersurface in a manner formally identical with the mode of definition of the co-efficients $F_j{}^i{}_k$ in terms of the fundamental tensor g_{ij} of F^n.

On the other hand, for the h(hv)-torsion tensor C_{ijk} of a Finsler space we have ([2])

(3.2)　$C_{ijk} = C_{\alpha\beta\gamma}B_i^{\alpha}B_j^{\beta}B_k^{\gamma} + M_{\alpha\beta}(B_i^{\alpha}B_j^{\beta}N_k + B_j^{\alpha}B_k^{\beta}N_i + B_k^{\alpha}B_i^{\beta}N_j)$
$\qquad + M_{\alpha}(B_i^{\alpha}N_jN_k + B_j^{\alpha}N_kN_i + B_k^{\alpha}N_iN_j) + MN_iN_jN_k$,

where $C_{\alpha\beta\gamma}$ is the projection of C_{ijk} onto the hypersurface, M is the normal components of C_{ijk} and

(3.3)　$M_{\alpha\beta} = C_{ijk}B_{\alpha\beta}^{ij}N^k$,　$M_{\alpha} = C_{ijk}B_{\alpha}^iN^jN^k$.

The tensor $M_{\alpha\beta}$ in (4.2) will be called a *Brown tensor* over a hypersurface of a Finsler space. Let us denote the difference of induced and intrinsic connection coefficients of a hypersurface by $\Lambda_{\beta}{}^{\alpha}{}_{\gamma}$ ([9]). From (3.1), we have

(3.4)　$\Lambda_{\beta}{}^{\alpha}{}_{\gamma} = \overline{F}_{\beta}{}^{\alpha}{}_{\gamma} - F_{\beta}{}^{\alpha}{}_{\gamma}$.

It is then shown [2] that

(3.5)　$\Lambda_{\beta\alpha\gamma}\dot{u}^{\beta} = NM_{\alpha\gamma}$,　$\Lambda_{\beta\alpha\gamma}\dot{u}^{\beta}\dot{u}^{\gamma} = M_{\alpha\gamma}\dot{u}^{\gamma} = 0$.

The following has been proved by Brown ((2]):

Lemma.2 *Assuming that $N \neq 0$, the induced and intrinsic connection coefficients coincide if and only if $M_{\alpha\beta} = 0$ over the Finsler hypersurface.*

§ 4. Normal unit vector of C-rebucible Finsler space.

In this section, we shall consider the normal unit vector of a C-reducible Finsler space which is defined by M. Matsumoto. [5].

Definition. A Finsler space $F^n (n \geqq 3)$ is called *C-reducible* if the h(hv)-torsion tensor C_{ijk} is written in the form

(4.1) $C_{ijk} = (h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n+1)$.

Remark. M. Matsumoto also indicated two certain metrics of a C-reducible Finsler space, namely Randers metric $(L = \alpha + \beta)$ and Kropina metric $(L = \alpha^2/\beta)$. Moreover, M. Matsumoto and S. Hōjo [6] have proved that the metric functions of C-reducible Finsler spaces are confined solely to the above metrics.

It is well-known ([10], [11]) that the h(hv)-tortion tensor C_{ijk} of a Kropina space and a Randers space is respectively given by

(4.2) $\overset{k}{C}_{ijk} = (h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)/2L$, $m_i = 1_i - \tau b_i$,

(4.3) $\overset{R}{C}_{ijk} = (h_{ij}L_k + h_{jk}L_i + h_{ki}L_j)/2L$, $L_i = (1+\mu)b_i - \mu 1_i$, $\mu = \alpha^{-1}\beta$.

Since $N_k y^k = 0$ and $N_k B_\beta^k = 0$, from (3.3), (4.2) and (4.3), the Brown tensor $M_{\alpha\beta}$ of a C-reducible Finsler space F^n is given by

(4.4) $M_{\alpha\beta} = N^k C_k h_{\alpha\beta}/(n+1)$.

On the other hand, the torsion vector $\overset{k}{C}_k$ of a Kropina space (resp. $\overset{R}{C}_k$ of a Randers space) is given by

(4.5) $\overset{k}{C}_k = (n+1)(1_k - \tau b_k)/2L$,   (resp. $\overset{R}{C}_k = (n+1)(\mu 1_k - \tau' b_k)/2L$,
          $\tau' = 1+\mu$), ([10], [11]).

Therefore, $M_{\alpha\beta}$ of a C-reducible Finsler space F^n reduces to

(4.4') $M_{\alpha\beta} = \nu b_j N^j h_{\alpha\beta}/2L$,

where ν is some scalar. Consequently, we have

Theorem 1. *Let the covariant vector field b_i be tangential to the hypersurface of a C-reducible Finsler space. Then the induced and intrinsic connection coincide over the Finsler hypersurface.*

§5. Induced and intrinsic connection parameters of Kropina hypersurface.

In virtue of (1.2), the induced connection parameters $F_\beta{}^\alpha{}_\gamma$ of a Kropina hypersurface H^{n-1} is written in the form

(5.1) $F_\beta{}^\alpha{}_\gamma = B_i^\alpha(B_{\beta\gamma}^i + \{^i_{jk}\}B_{\beta\gamma}^{jk}) + B_i^\alpha D_j{}^i{}_k B_{\beta\gamma}^{jk}.$

Since the induced and intrinsic Christoffel symbols of the associated Riemannian hypersurface R^{n-1} are equal, from (2.24) and (5.1) we have

(5.2) $F_\beta{}^\alpha{}_\gamma = \{_\beta{}^\alpha{}_\gamma\} + V_\beta{}^\alpha{}_\gamma + \Phi(b^\alpha - 2 1^\alpha)\bar\Omega_{\beta\gamma}/\bar\rho,$

where we put

(5.3)   (a) $V_\beta{}^\alpha{}_\gamma = B_i^\alpha D_j{}^i{}_k B_{\beta\gamma}^{jk},$

       (b) $\{_\beta{}^\alpha{}_\gamma\} = \overline{B}_i^\alpha(B_{\beta\gamma}^i + \{^i_{jk}\}B_{\beta\gamma}^{jk}),$

and $\bar\Omega_{\beta\gamma}$ are components of the second fundamental tensor of the Riemannian hypersurface R^{n-1}. Contraction of (5.3)a by $\dot u^\gamma$ yields

(5.4) $V_\beta{}^\alpha{}_\gamma \dot u^\gamma = B_i^\alpha D_j{}^i{}_0 B_\beta^j.$

The intrinsic connection parameters $\overline{F}_\beta{}^\alpha{}_\gamma$ of a Kropina hypersurface H^{n-1} are given by

(5.5) $\overline{F}_\beta{}^\alpha{}_\gamma = \{_\beta{}^\alpha{}_\gamma\} + D_\beta{}^\alpha{}_\gamma,$

where we put

(5.6) $D_\beta{}^\alpha{}_\gamma = -Q^{\alpha\epsilon}(F_{\epsilon\beta}1_\gamma + F_{\epsilon\gamma}1_\beta) - E_{\beta\gamma}Q^\alpha - h_\gamma^\alpha\Phi_\beta - h_\beta^\alpha\Phi_\gamma + h_{\beta\gamma}\Phi^\alpha + \bar\lambda C_\beta{}^\alpha{}_\gamma,$

and

$2E_{\alpha\beta} = b_{\alpha\beta} + b_{\beta\alpha}, \qquad 2F_{\alpha\beta} = b_{\alpha\beta} - b_{\beta\alpha},$

$b^\alpha = a^{\alpha\beta}b_\beta, \qquad a^{\alpha\beta}a_{\beta\gamma} = \delta_\gamma^\alpha, \qquad \bar\rho = a_{\alpha\beta}b^\alpha b^\beta,$

$Q^\alpha = (2 1^\alpha - b^\alpha)/\bar\rho, \qquad Q^{\alpha\epsilon} = (a^{\alpha\epsilon} + Q^\alpha b^\epsilon)/2,$

(5.7) $\Phi_\alpha = (\bar\rho F_{0'\alpha}/\beta - \bar\rho Q^\epsilon F_{\epsilon\alpha} + 2b_{0'\alpha}/\overline{L} + F_{\epsilon 0'}b^\epsilon 1_\alpha/\overline{L})/2\bar\rho,$

$g^{\alpha\epsilon}\Phi_\epsilon = \Phi^\alpha, \qquad \bar\lambda = (E_{0'0'}/\overline{L} + F_{\epsilon 0'}b^\epsilon)/\bar\rho, \qquad \Phi_\alpha \dot u^\alpha = \bar\lambda.$

The suffix "$0'$" means the contraction by $\dot u^\alpha$. Contracting (5.6) by $\dot u^\gamma$ and using the relations $C_\beta{}^\alpha{}_\gamma \dot u^\gamma = 0$, $h_\gamma^\alpha \dot u^\gamma = 0$ and (5.7) we obtain

(5.8) $D_\beta{}^\alpha{}_{0'} = -\{a^{\alpha\epsilon}(\overline{L}F_{\epsilon\beta} + F_{\epsilon 0'}1_\beta) + b^\epsilon(2 1^\alpha - b^\alpha)(\overline{L}F_{\epsilon\beta} + F_{\epsilon 0'}1_\beta)/\bar\rho\}/2$

$- E_{0'\beta}(2 1^\alpha - b^\alpha)/\bar\rho - \bar\lambda h_\beta^\alpha.$

Differentiating (2.8) covariantly with respect to u^β in the Riemannian hypersurface R^{n-1}, we get

(5.9) $\nabla_\beta b_\alpha = b_{\alpha\beta} = b_{ij}B_{\alpha\beta}^{ij} + b_i \overline{I}_{\alpha\beta}^i,$

where $\overline{I}_{\alpha\beta}^i$ $(= \nabla_\beta B_\alpha^i)$ is the normal curvature vector of R^{n-1}. Since the unit normal vector

of $\overline{R}^{n-1}$ is $\overline{N}^l$, (5.9) may be written as

(5.10) $b_{\alpha\beta} = b_{lj}B_{\alpha\beta}^{lj} + b_l\overline{N}^l\overline{\Omega}_{\alpha\beta}.$

From (5.7) and (5.9), we have

(5.11)   (1) $E_{\alpha\beta} = E_{lj}B_{\alpha\beta}^{lj} + b_l\overline{N}^l\overline{\Omega}_{\alpha\beta},$ (2) $F_{\alpha\beta} = F_{lj}B_{\alpha\beta}^{lj},$

where we have used the fact that $\overline{\Omega}_{\alpha\beta}$ is symmetric in α and β. Owing to (5.2) and (5.5) the difference $\Lambda_\beta{}^\alpha{}_\gamma$ of the induced and intrinsic connection coefficients of a Kropina hypersurface are given by

(5.12) $\Lambda_\beta{}^\alpha{}_\gamma = \overline{F}_\beta{}^\alpha{}_\gamma - F_\beta{}^\alpha{}_\gamma = D_\beta{}^\alpha{}_\gamma - V_\beta{}^\alpha{}_\gamma - \phi(b^\alpha - 2 1^\alpha)\overline{\Omega}_{\beta\gamma}/\overline{\rho}.$

Multiplying (5.12) by $\dot{u}^\gamma$, using (3.5), (5.4) and (5.8) we obtain $NM_{\alpha\beta}$ of the Kropina hypersurface H^{n-1}:

(5.13)
$$NM_{\alpha\beta} = (\lambda-\lambda')h_{\alpha\beta} - (2 1_\alpha - g_{\alpha\gamma}b^\gamma)\{b^\varepsilon(\overline{L}F_{\varepsilon\beta} - F_{0'\varepsilon}1_\beta)/2 + E_{0'\beta}\}/\overline{\rho}$$
$$- g_{\alpha\gamma}a^{\gamma\varepsilon}(\overline{L}F_{\varepsilon\beta} + F_{\varepsilon0'}1_\beta)/2 + (2 1_\alpha - g_{\alpha\gamma}B_j^\gamma b^j)\{b^r(LF_{rj}B_\beta^j + F_{r0}1_\beta)/2 + E_{j0}B_\beta^j\}/\rho$$
$$+ g_{\alpha\gamma}B_j^\gamma a^{lr}(LF_{rj}B_\beta^j + F_{r0}1_\beta)/2 + \phi(2 1_\alpha - g_{\alpha\gamma}b^\gamma)\overline{\Omega}_{\beta0'}/\overline{\rho}.$$

On direct calculation with the help of relations (1.1), (2.5), (2.6'), (2.8), (2.9), (2.19), (5.7) and (5.11), we get

(5.14)
$$2 1_\alpha - g_{\alpha\beta}b^\beta = \overline{\rho}\,\overline{\tau}(2 1_\alpha - \overline{\tau}b_\alpha), \quad 2 1_\alpha - g_{\alpha\beta}B_j^\beta b^j = \overline{\tau}\rho(2 1_\alpha - \tau b_\alpha),$$
$$b^\varepsilon F_{\varepsilon0'} = b^j F_{j0} - \phi F_{j0}\overline{N}^j, \quad \phi = b_j\overline{N}^j, \quad b_{0'0'} = E_{00} + \Phi\overline{\Omega}_{0'0'}.$$

Consequently, in virtue of (2.23), (2.24) and (5.14), $NM_{\alpha\beta}$ is written in the form

(5.13') $NM_{\alpha\beta} = (b_l\overline{N}^l)\{b_r\overline{N}^r(E_{00}/L + F_{r0}b^r)/\rho + \overline{\Omega}_{0'0'}/L - F_{r0}\overline{N}^r\}h_{\alpha\beta}.$

From (5.13') we have to discuss the two cases given by

(5.15)   (A) $b_l\overline{N}^l = 0,$ (B) $b_l\overline{N}^l \neq 0.$

First, we consider the case (A). In this case R^{n-1} is called a *tangential associated hypersurface*, because the covariant vector field b_i is tangential to the associated Riemannian hypersurface R^{n-1}. From (5.13') and lemma 2 we can state

Theorem 2. *On a tangential associated Riemannian hypersurface, the induced and intrinsic connections coincide with each other.*

For a tangential associated hypersurface R^{n-1}, from (5.10) we obtain $b_{\alpha\beta} = b_{lj}B_{\alpha\beta}^{lj}$, so that it follows that

(5.16) $b_{\alpha\beta}B_j^\alpha B_k^\beta = b_{hl}H_j^h H_k^l,$

where we put $H_{hj} = a_{hj} - \overline{N}_h\overline{N}_j$ and $H_j^h = a^{hm}H_{mj}$. Since $\phi = b_l\overline{N}^l = 0$, if $\nabla_j\overline{N}^l = 0$, (5.16) yields $b_{\alpha\beta}B_{jk}^{\alpha\beta} = b_{jk}$. Thus we have

Theorem 3. *Assume that an associated Riemannian hypersurface R^{n-1} be tangential and the unit normal vector fiejd $\bar{N}^i$ of R^{n-1} is parallel with respect to the associated Riemannian connection. Then $\nabla_j b_1 = 0$ if and only if $\nabla_\alpha b_\beta = 0$.*

In (5.10), if the vector field b_1 is parallel with respect to the associated Riemannian connection, that is $b_{ij} = 0$, then we get

(5.17) $b_{\alpha\beta} = b_1 \bar{N}^1 \bar{\Omega}_{\alpha\beta}$.

Here we can state

Theorem 4. *Assume that the covariant vector b_1 be parallel with respect to the associated Riemannian connection and the associated Riemannian hypersurface R^{n-1} be not totally geodesic. Then an associated Riemannian hypersurface R^{n-1} is tangential if and only if $\nabla_\alpha b_\beta = 0$.*

Definition. A Finsler space is called an *affinely connected* space if the Berwald connection coefficients are functions of position only, such a space will be called a *Berwald sqace.*

Lemma 3 ([11]). *If the covariant vector field b_i is parallel with respect to the associated Riemannian connection, then the Kropina space is the Berwald one.*

From (5.17) and the above lemma, we have

Theorem 5. *If the vector field b_1 is tangential to the Riemannian hypersurface R^{n-1}, then the Kropina hypersurface H^{n-1} is a Berwald space, provided that $b_{ij} = 0$.*

Next we consider the case $b_i \bar{N}^i \neq 0$. In virtue of (5.17), we have the following

Theorem 6. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection and $b_i \bar{N}^i \neq 0$. Then the associated hypersurface R^{n-1} is totally geodesic if and only if $\nabla_\alpha b_\beta = 0$.*

From the above theorem and the lemma 3, we obtain

Corollary. *Assume that the vector field b_1 be parallel with respect to the associated Riemannian connection and $b_i \bar{N}^1 \neq 0$. If the associated hypersurface R^{n-1} is totally geodesic, the Kropina hypersurface H^{n-1} is a Berwald space.*

Further from (5.13′) we get

Teeorem 7. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection. If the associated hypersurface R^{n-1} is totally geodesic, then the induced and intrinsic*

connections of a Kropina hypersurface coincide with each other, provided that $N \neq 0$.

Next, we assume that the vector field b_i is gradient, that is $2F_{ij} = b_{ij} - b_{ji} = 0$. Then (5.13′) yields

(5.18) $NM_{\alpha\beta} = (b_i \overline{N}^i)\{(b_r \overline{N}^r)b_{00}/\rho + \overline{\Omega}_{0'0'}\}/L.$

Here we get

Theorem 8. *Assume that the vector field b_i be gradient and $N \neq 0$, $b_i \overline{N}^i \neq 0$. Then the induced and intrinsic connections of a Kropina hypersurface coincide with each other if and only if the relation*

(5.19) $(b_i \overline{N}^i)b_{00}/\rho + \overline{\Omega}_{0'0'} = 0$

holds.

Also, the following lemma has been proved by Brown [2] :

Lemma 4. *A geodesic of a Finsler hypersurface is a geodesic of a Finsler space if and only if $N = \Omega_{\alpha\beta}\, \dot{u}^{\alpha}\dot{u}^{\beta} = 0$ along the curve, where $\Omega_{\alpha\beta}$ are to be considered as the components of the second fundamental tensor of the Finsler hypersurface.*

Using the above lemma and (5.18), we get

Theorem 9. *Assume that the vector field b_i be gradient and $M_{\alpha\beta} \neq 0$, $b_i \overline{N}^i \neq 0$. Then a geodesic of a Kropina hypersurface H^{n-1} is a geodesic of a Kropina space F^n if and only if the relation (5.19) holds.*

From the above and (5.17) we can state

Theorem 10. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection and $M_{\alpha\beta} \neq 0$, $b_i \overline{N}^i \neq 0$. If $\nabla_{\alpha}b_{\beta} = 0$, then a geodesic of the Kropina hypersurface H^{n-1} is a geodesic of a Kropina space F^n.*

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