クロピナ空間の超曲面の誘導的,内在的理論について

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On Induced and Intrinsic Theories of Hypersurfaces of Kropina Spaces.

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Abstract

Let $R^n = (M^n, \alpha)$ be an n-dimensional Riemannian space with a Riemannian metric $\alpha(x, dx) = (a_{ij}(x)dx^i dx^j)^{1/2}$ and let $F^n = (M^n, L)$ be a Kropina space with the fundamental function $L(x, y) = \alpha^2(x, y)/\beta(x, y)$, where $\beta(x, dx) = b_i(x)dx^i$.

The purpose of the present paper is to study the induced and intrinsic theories of hypersurface of a Kropina space.

§0. Introduction. The induced and intrinsic theories of the subspaces of a Finsler space have been studied by Davies ([3]) and Rund ([9]). The connection coefficients of a Kropina hypersurface can be written as the sum of Riemannian Christoffel symbols and other tensor. In this paper we compare the induced connection coefficients with intrinsic connection coefficients of a Kropina hypersurface and discuss whether they coincide or not. The notations and terminologies are refered to Matsumoto's monograph [7].

§1. Preliminaries. Let F^n be an n–dimensional Kropina space. Components g_{ij} of the fundamental tensor field are given by $g_{ij} = (\partial^2 L^2/\partial y^i \partial y^j)/2$, and the covariant components $y_i = g_{ij}y^j$ of the supporting element are given by $L\partial L/\partial y^i$. The angular metric tensor $h_{ij}(x,y)$ is defined as $h_{ij} = g_{ij} - l_i l_j$, $l_i = y_i/L$. The Riemannian space R^n with the metric $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ is called the *associated Riemannian space with F^n*. The Christoffel symbols of R^n are denoted by $\{^{\,i}_{jk}\}$ and this Riemannian connection is called the *associated one*. We denote by ∇_k the covariant differentiation with respect to x_k relative to the associated Riemannian connection. The fundamental tensor g_{ij} and the connection coefficients $F_j{}^i{}_k$ of the Cartan connection are given respectively by ([11]).

(1.1) $$g_{ij} = \tau(2a_{ij} - l_i b_j - l_j b_i) + l_i l_j, \qquad \tau = \alpha^2\beta^{-2}$$

(1.2) $$F_j{}^i{}_k = \{{}_j{}^i{}_k\} + D_j{}^i{}_k.$$

The tensor D^i_{jk}, called the *difference tensor*, is given by ([11])

(1.3) $$D_j{}^i{}_k = -Q^{ir}(F_{rj}l_k + F_{rk}l_j) - E_{jk}Q^i - h^i{}_j\Phi_k$$
$$-h^i{}_k\Phi_j + h_{jk}\Phi^i + \lambda C_j{}^i{}_k,$$

where we put

(1.4)
$$\begin{aligned}
&(1)\quad \nabla_k b_j = b_{jk}, \quad 2E_{jk} = b_{jk} + b_{kj}, \quad 2F_{jk} = b_{jk} - b_{kj},\\
&(2)\quad b^i = a^{ij}b_j, \quad a^{ij}a_{jk} = \delta^i{}_k, \quad \rho = a_{ij}b^ib^j,\\
&\qquad Q^i = (2l^i - b^i)/\rho, \quad Q^{ir} = (a^{ir} + Q^ib^r)/2,\\
&(3)\quad \Phi_k = (\rho F_{0k}/\beta - \rho Q^r F_{rk} + 2b_{0k}/L + F_{r0}b^r l_k/L)/2,\\
&\qquad g^{ir}\Phi_r = \Phi^i, \quad \lambda = (E_{00}/L + F_{r0}b^r)/\rho, \quad \Phi_k y^k = \lambda.
\end{aligned}$$

In (1.4) 3) and the remainder of the present paper the suffix " 0 " means the contraction by y^i. Contraction of (1.3) by y^k gives

(1.5) $$D_j{}^i{}_0 = -\{a^{ir}(LF_{rj} + F_{r0}l_j) + b^r(2l^i - b^i)(LF_{rj} +$$
$$+F_{r0}l_j)/\rho\}/2 - E_{j0}(2l^i - b^i)/\rho - \lambda h^i{}_j,$$

where (1.4) was used.

Lemma 1([11]). *The difference tensor D^i_{jk} vanishes if and only if the covariant vector b_i is parallel with respect to the associated Riemannian connection, i.e., $\nabla_k b_i = 0$.*

§ 2. Hypersurfaces of Kropina space and associated Riemannian space.

First, we are concerned with a hypersurface H^{n-1} of the underlying manifold M^n of a Kropina space $F^n = (M^n, L)$, which is represented parametrically by

(2.1) $$x^i = x^i(u^\sigma), \qquad \sigma = 1, 2, \cdots, n-1,$$

where u^σ are Gaussian coordinates on H^{n-1}. Introducing the notations

(2.2) $$B^i_\alpha = \partial x^i/\partial u^\alpha,$$

we shall assume that the matrix of these projection factors is of rank n−1. The following notations are also employed:

(2.3) $B^i_{\alpha\beta} = \partial^2 x^i/\partial u^\alpha \partial u^\beta$, $B^{ij\cdots k}_{\alpha\beta\cdots\gamma} = B^i_\alpha B^j_\beta \cdots B^k_\gamma$.

The functions $B^i_\alpha(x)$ may be considered as components of $n-1$ linearly independent vectors tangent to H^{n-1}. Therefore any vector $\dot{x}^i$, tangent to H^{n-1}, may be written uniquely in the form

(2.4) $x^i = B^i_\alpha x^\alpha$,

where X^α are components of the vector relative to the u-coordinate system. In particular, we assume that the supporting element y^i is tangential to H^{n-1} so that

(2.5) $y^i(=\dot{x}^i) = B^i_\alpha \dot{u}^\alpha$.

The induced fundamental metric tensor $g_{\alpha\beta}(u, \dot{u})$ of the hypersurface H^{n-1} defined with respect to such a direction is given by

(2.6) $g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, y) B^{ij}_{\alpha\beta}$.

If $L(x, y)$ represents the fundamental function of F^n for a direction y^i tangent to H^{n-1}, it follows from (2.5) that the corresponding fundamental function for H^{n-1} is given by $\overline{L}(u, \dot{u}) = L(x^i(u), B^i_\alpha \dot{u}^\alpha)$. For the Kropina space F^n, it follows from (2.1) and (2.5) that the fundamental function $\overline{L}$ is given by

(2.7) $\overline{L}(u, \dot{u}) = a_{\alpha\beta}(u)\dot{u}^\alpha \dot{u}^\beta / b_\beta \dot{u}^\beta$, $a_{\alpha\beta} = a_{ij} B^{ij}_{\alpha\beta}$,

in which $a_{\alpha\beta}(u)$ is the fundamental tensor of the Riemannian hypersurface R^{n-1} and $b_\alpha(u)$ is given by

(2.8) $b_\alpha = b_i B^i_\alpha$.

Thus, in virtue of (1.1), (2.7) and (2.8), the induced metric tensor $g_{\alpha\beta}$ in (2.6) is written by

(2.6′) $g_{\alpha\beta} = \overline{\tau}(2a_{\alpha\beta} - l_\alpha b_\beta - l_\beta b_\alpha) + l_\alpha l_\beta$, $\overline{\tau} = \tau$.

Here we have

Proposition 1. *A hypersurface of a Kropina space is also a Kropina space.*

Remark. From the above proposition, the hypersurface of a Kropina space is called a *Kropina hypersurface.*

Further, we have

(2.9) $l^i = B^i_\alpha l^\alpha$.

As usual, $\det(g_{ij}) \neq 0$ is supposed. Thus according to our assumption the tensor $g_{\alpha\beta}(u, \dot{u})$ possesses the reciprocal tensor $g_{\alpha\beta}$ which is used to define a set of $n-1$ covariant vectors

(2.10) $B^\alpha_i(x, y) = g^{\alpha\beta}(u, \dot{u}) g_{ij}(x, y) B^j_\beta(x)$, which satisfy

(2.11) $B^i_\alpha B^\beta_i = \delta^\beta_\alpha$.

Another useful indentity ([3]) is

(2.12) $B^\alpha_i B^j_\alpha = \delta^j_i - N_i N^j$,

where the unit normal vector $N^i(x, y)$ is defined at each point of the Kropina hypersurface H^{n-1} with respect to the tangential supporting element y^i by a system of equations

(2.13) $N^i = g^{ij}(x, y) N_j, \quad g_{ij} N^i N^j = 1, \quad N_i B^i_\alpha = 0$,

which in turn imply

(2.14) $N^i B^\alpha_i = 0$.

Further we get

(2.15) $g_{ij} = g_{\alpha\beta} B^\alpha_i B^\beta_j + N_i N_j, \qquad g^{ij} = g^{\alpha\beta} B^i_\alpha B^j_\beta + N^j N^i$.

Next, we shall consider a hypersurface R^{n-1} of the associated Riemannian space with the metric $\alpha = (a_{ij}(x) y^i y^j)^{1/2}$ represented parametrically by the same equations as (2.1). Then u^α in (2.1) are Gaussian coordinates on R^{n-1}. And the function $B^i_\alpha(x)$ in (2.2) may be considered to be components of a set of $n-1$ linearly independent vectors tangent to R^{n-1}. The induced fundamental metric tensor of the Riemannian hypersurface R^{n-1} is given by $a_{\alpha\beta}$ in (2.7). The hypersurface of the associated Riemannian space R^n is called an *associated Riemannian hypersurface* $R^{n-1} = (M^{n-1}, \ \bar{\alpha} = (a_{\alpha\beta}(u)\dot{u}^\alpha \dot{u}^\beta)^{1/2})$.
The quantities $\bar{B}^\alpha_i(x)$ are uniquely defined along R^{n-1} by the equations

(2.16) $\bar{B}^\alpha_i(x) = a^{\alpha\beta}(u) a_{ij} B^j_\beta(x)$.

We denote the covariant components of a unit normal vector of R^{n-1} by $\bar{N}^i$. Then we have a field of linear frame $(B^i_1, \cdots, B^i_{n-1}; \ \bar{N}^i = a^{ij}\bar{N}_j)$ of R^n defined along R^{n-1} by

(2.17) $B^i_\alpha \bar{B}^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha \bar{B}^\alpha_j = \delta^i_j - \bar{N}^i \bar{N}_j,, \qquad \bar{N}^i \bar{B}^\alpha_i = 0$.

It follows from (2.17) that

(2.18) $a_{ij} = a_{\alpha\beta} \bar{B}^\alpha_i \bar{B}^\beta_j + \bar{N}_i \bar{N}_j$

Since $\bar{N}_i B^i_\alpha = 0$ and $B^i_\alpha \dot{u}^\alpha = y^i$, we see that the supporting element y^i is tangential to the associated Riemannian hypersurface R^{n-1}, that is $\bar{N}_i y^i = 0$, so that we have

(2.19) $\bar{N}^i Y_i = 0, \qquad Y_i = a_{ij} y^j$,

which will play an important role later on. The reciprocal tensor $g^{\alpha\beta}$ of $g_{\alpha\beta}$ is given by

(2.20) $g^{\alpha\beta} = [\bar{\rho} a^{\alpha\beta} + 2(l^\alpha b^\beta + l^\beta b^\alpha) - b^\alpha b^\beta + 2(\bar{\rho}\bar{\tau} - 2) l^\alpha l^\beta] / 2\bar{\rho}\bar{\tau}$,

where we put

(2.21) (a) $\bar{\rho} = a_{\alpha\beta}b^\alpha b^\beta$, (b) $a^{\alpha\beta}b_\alpha = b^\beta$, (c) $1^\alpha = g^{\alpha\beta}1_\beta$.

With the help of relations (1.4)2), (2.7), (2.8) and (2.21)a), we can easily obtain

(2.22) $\bar{\rho} = \rho - (b_i\overline{N}^i)^2$.

It follows from (2.6′), (2.20), (2.21) and (2.22) that

(2.23)
(1) $Y^\alpha = (\dot{u}^\alpha)/\beta = \tau 1^\alpha$, (2) $1_\alpha b^\alpha = 2 - \bar{\rho}\tau$,
(3) $a^{\alpha\varepsilon}1_\varepsilon = 2Y^\alpha - \tau b^\alpha$, (4) $b^j = b^\alpha B^j_\alpha + (b_i\overline{N}^i)\overline{N}^j$.

Further, in virtue of (2.10), (2.16) and (2.23) we have

(2.24) $B^\alpha_i = \overline{B}^\alpha_i + (b_m\overline{N}^m)(b^\alpha - 21^\alpha)\overline{N}_i/\bar{\rho}$.

§ 3. Relation between induced and intrinsic connection parameters.

The Cartan connection coefficients of the Finsler space F^n are denoted by $F_j{}^i{}_k$. The induced connection parameters of hypersurface are defined by the relation ([8])

(3.1) $F_\gamma{}^\alpha{}_\beta = B^\alpha_i(B^i_{\beta\gamma} + F_j{}^i{}_k B^{jk}_{\beta\gamma})$.

And the intrinsic connection coefficients $\overline{F}_\beta{}^\alpha{}_\gamma$ are defined with respect to the induced metric (2.6) of hypersurface in a manner formally identical with the mode of definition of the coefficients $F_j{}^i{}_k$ in terms of the fundamental tensor g_{ij} of F^n.

On the other hand, for the h(hv)-torsion tensor C_{ijk} of a Finsler space we have ([2])

(3.2) $C_{ijk} = C_{\alpha\beta\gamma}B^\alpha_i B^\beta_j B^\gamma_k + M_{\alpha\beta}(B^\alpha_i B^\beta_j N_k + B^\alpha_j B^\beta_k N_i + B^\alpha_k B^\beta_i N_j)$
$+ M_\alpha(B^\alpha_i N_j N_k + B^\alpha_j N_k N_i + B^\alpha_k N_i N_j) + MN_iN_jN_k$,

where $C_{\alpha\beta\gamma}$ is the projection of C_{ijk} onto the hypersurface, M is the normal components of C_{ijk} and

(3.3) $M_{\alpha\beta} = C_{ijk}B^{ij}_{\alpha\beta}N^k$, $M_\alpha = C_{ijk}B^i_\alpha N^j N^k$.

The tensor $M_{\alpha\beta}$ in (4.2) will be called a *Brown tensor* over a hypersurface of a Finsler space. Let us denote the difference of induced and intrinsic connection coefficients of a hypersurface by $\Lambda_\beta{}^\alpha{}_\gamma$ ([9]). From (3.1), we have

(3.4) $\Lambda_\beta{}^\alpha{}_\gamma = \overline{F}_\beta{}^\alpha{}_\gamma - F_\beta{}^\alpha{}_\gamma$.

It is then shown [2] that

(3.5) $\Lambda_{\beta\alpha\gamma}\dot{u}^\beta = NM_{\alpha\gamma}$, $\Lambda_{\beta\alpha\gamma}\dot{u}^\beta\dot{u}^\gamma = M_{\alpha\gamma}\dot{u}^\gamma = 0$.

The following has been proved by Brown ([2]):

Lemma.2 *Assuming that $N \neq 0$, the induced and intrinsic connection coefficients coincide if and only if $M_{\alpha\beta} = 0$ over the Finsler hypersurface.*

§ 4. Normal unit vector of C-rebucible Finsler space.

In this section, we shall consider the normal unit vector of a C-reducible Finsler space which is defined by M. Matsumoto. [5].

Definition. A Finsler space $F^n (n \geqq 3)$ is called *C-reducible* if the h(hv)-torsion tensor C_{ijk} is written in the form

(4.1) $$C_{ijk} = (h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n+1).$$

Remark. M. Matsumoto also indicated two certain metrics of a C-reducible Finsler space, namely Randers metric $(L = \alpha + \beta)$ and Kropina metric $(L = \alpha^2/\beta)$. Moreover, M. Matsumoto and S. Hōjō [6] have proved that the metric functions of C-reducible Finsler spaces are confined solely to the above metrics.

It is well-known ([10], [11]) that the h(hv)-tortion tensor C_{ijk} of a Kropina space and a Randers space is respectively given by

(4.2) $$\overset{k}{C}_{ijk} = (h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)/2L, \qquad m_i = l_i - \tau b_i,$$

(4.3) $$\overset{R}{C}_{ijk} = (h_{ij}L_k + h_{jk}L_i + h_{ki}L_j)/2L, \qquad L_i = (1+\mu)b_i - \mu l_i, \qquad \mu = \alpha^{-1}\beta.$$

Since $N_k y^k = 0$ and $N_k B^k_\beta = 0$, from (3.3), (4.2) and (4.3), the Brown tensor $M_{\alpha\beta}$ of a C-reducible Finsler space F^n is given by

(4.4) $$M_{\alpha\beta} = N^k C_k h_{\alpha\beta}/(n+1).$$

On the other hand, the torsion vector $\overset{k}{C}_k$ of a Kropina space (resp. $\overset{R}{C}_k$ of a Randers space) is given by

(4.5) $$\overset{k}{C}_k = (n+1)(l_k - \tau b_k)/2L, \qquad (\text{resp. } \overset{R}{C}_k = (n+1)(\mu l_k - \tau' b_k)/2L,$$
$$\tau' = 1+\mu), \ ([10], [11]).$$

Therefore, $M_{\alpha\beta}$ of a C-reducible Finsler space F^n reduces to

(4.4′) $$M_{\alpha\beta} = \nu b_j N^j h_{\alpha\beta}/2L,$$

where ν is some scalar. Consequently, we have

Theorem 1. *Let the covariant vector field b_i be tangential to the hypersurface of a C-reducible Finsler space. Then the induced and intrinsic connection coincide over the Finsler hypersurface.*

§5. Induced and intrinsic connection parameters of Kropina hypersurface.

In virtue of (1.2), the induced connection parameters $F_\beta{}^\alpha{}_\gamma$ of a Kropina hypersurface H^{n-1} is written in the form

$$F_\beta{}^\alpha{}_\gamma = B^\alpha_i(B^i_{\beta\gamma}+\{{}_j{}^i{}_k\}B^{jk}_{\beta\gamma})+B^\alpha_i D_j{}^i{}_k B^{jk}_{\beta\gamma}. \tag{5.1}$$

Since the induced and intrinsic Christoffel symbols of the associated Riemannian hypersurface R^{n-1} are equal, from (2.24) and (5.1) we have

$$F_\beta{}^\alpha{}_\gamma = \{{}_\beta{}^\alpha{}_\gamma\}+V_\beta{}^\alpha{}_\gamma+\Phi(b^\alpha-2l^\alpha)\bar{\Omega}_{\beta\gamma}/\bar{\rho}, \tag{5.2}$$

where we put

$$\text{(5.3)}\quad \text{(a)}\quad V_\beta{}^\alpha{}_\gamma = B^\alpha_i D_j{}^i{}_k B^{jk}_{\beta\gamma},$$
$$\text{(b)}\quad \{{}_\beta{}^\alpha{}_\gamma\} = \bar{B}^\alpha_i(B^i_{\beta\gamma}+\{{}_j{}^i{}_k\}B^{jk}_{\beta\gamma}),$$

and $\bar{\Omega}_{\beta\gamma}$ are components of the second fundamental tensor of the Riemannian hypersurface R^{n-1}. Contraction of (5.3)a) by $\dot{u}^\gamma$ yields

$$V_\beta{}^\alpha{}_\gamma\dot{u}^\gamma = B^\alpha_i D_j{}^i{}_0 B^j_\beta. \tag{5.4}$$

The intrinsic connection parameters $\bar{F}_\beta{}^\alpha{}_\gamma$ of a Kropina hypersurface H^{n-1} are given by

$$\bar{F}_\beta{}^\alpha{}_\gamma = \{{}_\beta{}^\alpha{}_\gamma\}+D_\beta{}^\alpha{}_\gamma, \tag{5.5}$$

where we put

$$D_\beta{}^\alpha{}_\gamma = -Q^{\alpha\varepsilon}(F_{\varepsilon\beta}l_\gamma+F_{\varepsilon\gamma}l_\beta)-E_{\beta\gamma}Q^\alpha-h^\alpha_\gamma\Phi_\beta-h^\alpha_\beta\Phi_\gamma+h_{\beta\gamma}\Phi^\alpha+\bar{\lambda}C_\beta{}^\alpha{}_\gamma, \tag{5.6}$$

and

$$2E_{\alpha\beta} = b_{\alpha\beta}+b_{\beta\alpha},\qquad 2F_{\alpha\beta} = b_{\alpha\beta}-b_{\beta\alpha},$$
$$b^\alpha = a^{\alpha\beta}b_\beta,\qquad a^{\alpha\beta}a_{\beta\gamma} = \delta^\alpha_\gamma,\qquad \bar{\rho} = a_{\alpha\beta}b^\alpha b^\beta,$$
$$Q^\alpha = (2l^\alpha-b^\alpha)/\bar{\rho},\qquad Q^{\alpha\varepsilon} = (a^{\alpha\varepsilon}+Q^\alpha b^\varepsilon)/2,$$
$$\text{(5.7)}\quad \Phi_\alpha = (\bar{\rho}F_{0'\alpha}/\beta-\bar{\rho}Q^\varepsilon F_{\varepsilon\alpha}+2b_{0'\alpha}/\bar{L}+F_{\varepsilon 0'}b^\varepsilon l_\alpha/\bar{L})/2\bar{\rho},$$
$$g^{\alpha\varepsilon}\Phi_\varepsilon = \Phi^\alpha,\quad \bar{\lambda} = (E_{0'0'}/\bar{L}+F_{\varepsilon 0'}b^\varepsilon)/\bar{\rho},\quad \Phi_\alpha\dot{u}^\alpha = \bar{\lambda}.$$

The suffix "0′" means the contraction by $\dot{u}^\alpha$. Contracting (5.6) by $\dot{u}^\gamma$ and using the relations $C_\beta{}^\alpha{}_\gamma\dot{u}^\gamma = 0$, $h_\gamma{}^\alpha\dot{u}^\gamma = 0$ and (5.7) we obtain

$$\text{(5.8)}\quad D_\beta{}^\alpha{}_{0'} = -\{a^{\alpha\varepsilon}(\bar{L}F_{\varepsilon\beta}+F_{\varepsilon 0'}l_\beta)+b^\varepsilon(2l^\alpha-b^\alpha)(\bar{L}F_{\varepsilon\beta}+F_{\varepsilon 0'}l_\beta)/\bar{\rho}\}/2$$
$$-E_{0'\beta}(2l^\alpha-b^\alpha)/\bar{\rho}-\bar{\lambda}h^\alpha_\beta.$$

Differentiating (2.8) covariantly with respect to u^β in the Riemannian hypersurface R^{n-1}, we get

$$\nabla_\beta b_\alpha = b_{\alpha\beta} = b_{ij}B^{ij}_{\alpha\beta}+b_i\bar{I}^i_{\alpha\beta}, \tag{5.9}$$

where $\bar{I}^i_{\alpha\beta}$ $(= \nabla_\beta B^i_\alpha)$ is the normal curvature vector of R^{n-1}. Since the unit normal vector of $\overline{R}^{n-1}$ is $\overline{N}^i$, (5.9) may be written as

(5.10) $$b_{\alpha\beta} = b_{ij}B^{ij}_{\alpha\beta} + b_i\overline{N}^i\overline{\Omega}_{\alpha\beta}.$$

From (5.7) and (5.9), we have

(5.11) $$(1)\quad E_{\alpha\beta} = E_{ij}B^{ij}_{\alpha\beta} + b_i\overline{N}^i\overline{\Omega}_{\alpha\beta}, \qquad (2)\quad F_{\alpha\beta} = F_{ij}B^{ij}_{\alpha\beta},$$

where we have used the fact that $\overline{\Omega}_{\alpha\beta}$ is symmetric in α and β. Owing to (5.2) and (5.5) the difference $\Lambda_\beta{}^\alpha{}_\gamma$ of the induced and intrinsic connection coefficients of a Kropina hypersurface are given by

(5.12) $$\Lambda_\beta{}^\alpha{}_\gamma = \overline{F}_\beta{}^\alpha{}_\gamma - F_\beta{}^\alpha{}_\gamma = D_\beta{}^\alpha{}_\gamma - V_\beta{}^\alpha{}_\gamma - \phi(b^\alpha - 21^\alpha)\overline{\Omega}_{\beta\gamma}/\bar{\rho}.$$

Multiplying (5.12) by $\dot{u}^\gamma$, using (3.5), (5.4) and (5.8) we obtain $NM_{\alpha\beta}$ of the Kropina hypersurface H^{n-1}:

(5.13) $$\begin{aligned} NM_{\alpha\beta} &= (\lambda - \lambda')h_{\alpha\beta} - (21_\alpha - g_{\alpha\gamma}b^\gamma)\{b^\varepsilon(\overline{L}F_{\varepsilon\beta} - F_{0'\varepsilon}l_\beta)/2 + E_{0'\beta}\}/\bar{\rho} \\ &\quad - g_{\alpha\gamma}a^{\gamma\varepsilon}(\overline{L}F_{\varepsilon\beta} + F_{\varepsilon 0'}l_\beta)/2 + (21_\alpha - g_{\alpha\gamma}B^\gamma_i b^i)\{b^r(LF_{rj}B^j_\beta + F_{r0}l_\beta)/2 + E_{j0}B^j_\beta\}/\rho \\ &\quad + g_{\alpha\gamma}B^\gamma_i a^{ir}(LF_{rj}B^j_\beta + F_{r0}l_\beta)/2 + \phi(21_\alpha - g_{\alpha\gamma}b^\gamma)\overline{\Omega}_{\beta 0'}/\bar{\rho}. \end{aligned}$$

On direct calculation with the help of relations (1.1), (2.5), (2.6′), (2.8), (2.9), (2.19), (5.7) and (5.11), we get

(5.14) $$\begin{aligned} &21_\alpha - g_{\alpha\beta}b^\beta = \bar{\rho}\bar{\tau}(21_\alpha - \bar{\tau}b_\alpha), \quad 21_\alpha - g_{\alpha\beta}B^\beta_j b^j = \bar{\tau}\rho(21_\alpha - \tau b_\alpha), \\ &b^\varepsilon F_{\varepsilon 0'} = b^j F_{j0} - \phi F_{j0}\overline{N}^j, \quad \phi = b_j\overline{N}^j, \quad b_{0'0'} = E_{00} + \Phi\overline{\Omega}_{0'0'}. \end{aligned}$$

Consequently, in virtue of (2.23), (2.24) and (5.14), $NM_{\alpha\beta}$ is written in the form

(5.13′) $$NM_{\alpha\beta} = (b_i\overline{N}^i)\{b_r\overline{N}^r(E_{00}/L + F_{r0}b^r)/\rho + \overline{\Omega}_{0'0'}/L - F_{r0}\overline{N}^r\}h_{\alpha\beta}.$$

From (5.13′) we have to discuss the two cases given by

(5.15) $$(A)\quad b_i\overline{N}^i = 0, \qquad (B)\quad b_i\overline{N}^i \neq 0.$$

First, we consider the case (A). In this case R^{n-1} is called a *tangential associated hypersurface*, because the covariant vector field b_i is tangential to the associated Riemannian hypersurface R^{n-1}. From (5.13′) and lemma 2 we can state

Theorem 2. *On a tangential associated Riemannian hypersurface, the induced and intrinsic connections coincide with each other.*

For a tangential associated hypersurface R^{n-1}, from (5.10) we obtain $b_{\alpha\beta} = b_{ij}B^{ij}_{\alpha\beta}$, so that it follows that

(5.16) $$b_{\alpha\beta}B^\alpha_j B^\beta_k = b_{hi}H^h_j H^i_k,$$

where we put $H_{hj} = a_{hj} - \overline{N}_h\overline{N}_j$ and $H^h_j = a^{hm}H_{mj}$. Since $\phi = b_i\overline{N}^i = 0$, if $\nabla_j\overline{N}^i = 0$, (5.16) yields $b_{\alpha\beta}B^{\alpha\beta}_{jk} = b_{jk}$. Thus we have

Theorem 3. *Assume that an associated Riemannian hypersurface R^{n-1} be tangential and the unit normal vector fiejd $\bar{N}^i$ of R^{n-1} is parallel with respect to the associated Riemannian connection. Then* $\nabla_j b_i = 0$ *if and only if* $\nabla_\alpha b_\beta = 0$.

In (5.10), if the vector field b_i is parallel with respect to the associated Riemannian connection, that is $b_{ij} = 0$, then we get

(5.17) $$b_{\alpha\beta} = b_i \bar{N}^i \bar{\Omega}_{\alpha\beta}.$$

Here we can state

Theorem 4. *Assume that the covariant vector* b_i *be parallel with respect to the associated Riemannian connection and the associated Riemannian hypersurface R^{n-1} be not totally geodesic. Then an associated Riemannian hypersurface R^{n-1} is tangential if and only if $\nabla_\alpha b_\beta = 0$.*

Definition. A Finsler space is called an *affinely connected* space if the Berwald connection coefficients are functions of position only, such a space will be called a *Berwald sqace.*

Lemma 3 ([11]). *If the covariant vector field b_i is parallel with respect to the associated Riemannian connection, then the Kropina space is the Berwald one.*

From (5.17) and the above lemma, we have

Theorem 5. *If the vector field b_i is tangential to the Riemannian hypersurface R^{n-1}, then the Kropina hypersurface H^{n-1} is a Berwald space, provided that $b_{ij} = 0$.*

Next we consider the case $b_i \bar{N}^i \neq 0$. In virtue of (5.17), we have the following

Theorem 6. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection and $b_i \bar{N}^i \neq 0$. Then the associated hypersurface R^{n-1} is totally geodesic if and only if* $\nabla_\alpha b_\beta = 0$.

From the above theorem and the lemma 3, we obtain

Corollary. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection and* $b_i \bar{N}^i \neq 0$. *If the associated hypersurface R^{n-1} is totally geodesic, the Kropina hypersurface H^{n-1} is a Berwald space.*

Further from (5.13′) we get

Teeorem 7. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection. If the associated hypersurface R^{n-1} is totally geodesic, then the induced and intrinsic* *connections of a Kropina hypersurface coincide with each other, provided that* $N \neq 0$.

Next, we assume that the vector field b_i is gradient, that is $2F_{ij} = b_{ij} - b_{ji} = 0$. Then (5.13′) yields

(5.18) $\quad NM_{\alpha\beta} = (b_i\bar{N}^i)\{(b_r\bar{N}^r)b_{00}/\rho + \bar{\Omega}_{0'0'}\}/L.$

Here we get

Theorem 8. *Assume that the vector field* b_i *be gradient and* $N \neq 0$, $b_i\bar{N}^i \neq 0$. *Then the induced and intrinsic connections of a Kropina hypersurface coincide with each other if and only if the relation*

(5.19) $\quad (b_i\bar{N}^i)b_{00}/\rho + \bar{\Omega}_{0'0'} = 0$

holds.

Also, the following lemma has been proved by Brown [2] :

Lemma 4. *A geodesic of a Finsler hypersurface is a geodesic of a Finsler space if and only if* $N = \Omega_{\alpha\beta}\, \dot{u}^\alpha \dot{u}^\beta = 0$ *along the curve, where* $\Omega_{\alpha\beta}$ *are to be considered as the components of the second fundamental tensor of the Finsler hypersurface.*

Using the above lemma and (5.18), we get

Theorem 9. *Assume that the vector field* b_i *be gradient and* $M_{\alpha\beta} \neq 0$, $b_i\bar{N}^i \neq 0$. *Then a geodesic of a Kropina hypersurface* H^{n-1} *is a geodesic of a Kropina space* F^n *if and only if the* relation (5.19) holds.

From the above and (5.17) we can state

Theorem 10. *Assume that the vector field* b_i *be parallel with respect to the associated Riemannian connection and* $M_{\alpha\beta} \neq 0$, $b_i\bar{N}^i \neq 0$. *If* $\nabla_\alpha b_\beta = 0$, *then a geodesic of the Kropina hypersurface* H^{n-1} *is a geodesic of a Kropina space* F^n.

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References

[1] Berwald L. (1941), On Finsler and Cartan geometries III. Two dimensional Finsler spaces with rectilinear extremals, *Ann. of Math.* 2, 42, p. 84–112.

[2] Brown G.M. (1968), A study of tensors which charactrize a hypersurface of a Finsler space, *Canad. J. Math.*, 20 p. 1025–1036.

[3] Davies E.T. (1945), Subspaces of Finsler spaces, *Proc. Lond. Math. Soc.*, 49, p. 19–39.

[4] Kropina V. K. (1961), Projective two dimensional Finsler spaces with special metric, *Trudy Sem. Vector Tenzor Anal.*, 11, P. 277–292.

[5] Matsumoto M. (1972), On C-reducible Finsler spaces, *Tensor, N. S.*, 24, P. 29–37.

[6] Matsumoto M. and Hōjō S. (1978), A conclusive theorem on C-reducible Finsler spaces, *Tensor, N. S.*, 32, P. 225–230

[7] Matsumoto M. (to appear), Foundation of Finsler geometry and special Finsler spaces.

[8] Rund H. (1959), The differential geometry of Finsler spaces, *Springer Verlag Berlin,* P. 298.

[9] Rund N. (1965), The intrinsic and induced curvature theories of subspace of a Finsler space, *Tensor, N. S.*, 16, P. 294–312.

[10] Shibata C., Shimada H., Azuma M. and Yasuda H. (1977), On Finsler spaces with Randers' metric, *Tensor, N. S.*, 31, P. 219–226.

[11] Shibata C. (1978), On Finsler spaces with Kropina metric, *Reports on Math. phy.*, 13, P. 117–128.

[12] Shibata C. (to appear), On invariant tensors of β-changes of Finsler metrics, *J. Math. of Kyoto Univ.*