（α , β ）-計量をもつフィンスラー空間について

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On Finsler spaces with an (α, β)-metric

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柴　田　銕　光：(α, β)－計量をもつフィンスラー空間について

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Abstract

A Finsler metric $L(x, y)$ is called an (α, β)-*metric* if L is a positive-homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form.

The purpose of the present paper is to study the hv-curvature tensor P_{hijk} and the h-curvature tensor R_{hijk} of Finsler spaces with an (α, β)-metric.

§ 1. Introduction.

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, that is, an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. The concept of an (α, β)-metric was introduced in 1972 by Matsumoto [8]. A Finsler metric $L(x, y)$ is called an (α, β)-*metric* if L is a positive-homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form on M^n. We have specially interesting examples of an (α, β)-metric, for instance, $L = \alpha + \beta$ (Randers metric), $L = \alpha^2/\beta$ (Kropina metric), $L = (\alpha^2\beta)^{1/3}$ (a decomposition of cubic metric), and $L = \alpha^{m+1}/\beta^m$ (generalized Kropina metric). In particular, the Randers metric and the Kropina metric are of special interest in physics. The important role of the Randers metric in applied physics is conditioned by the fact that the Lagrangian function of the test charge in the gravitational and electromagnetic fields is the Randers metric $(n=4)$ with the Riemannian metric tensor of the time-space signature ([1],[2],[6]). At the same time, an interesting relation between the Kropina metric and the Lagrangian function of dynamical systems has been demonstrated by the present author ([20]).

Throughout the present paper we shall use the terminology and notations in Matsumoto's monograph [11].

§ 2. Cartan's connections.

In the present paper, we shall consider the Finsler space F^n with an (α,β)-metric. It is well known [8] that the angular metric tensor $h_{ij}=L\partial^2 L/\partial y^i\partial y^j$ is written as

$$h_{ij}=pa_{ij}+q_0 b_i b_j+q_{-1}(b_i Y_j+b_j Y_i)+q_{-2} Y_i Y_j, \tag{2.1}$$

where we put

$$\begin{aligned} &Y_i=a_{ij}y^j, \qquad p=LL_\alpha\alpha^{-1}, \qquad q_0=LL_{\beta\beta}, \\ &q_{-1}=LL_{\alpha\beta}\ \alpha^{-1}, \qquad q_{-2}=L\alpha^{-2}(L_{\alpha\alpha}-L_\alpha\alpha^{-1}). \end{aligned} \tag{2.1)1}$$

The fundamental tensor $g_{ij}=(\partial^2 L^2/\partial y^j\partial y^i)/2$ is given by

$$g_{ij}=pa_{ij}+p_0 b_i b_j+p_{-1}(b_i Y_j+b_j Y_i)+p_{-2} Y_i Y_j, \tag{2.2}$$

where we put

$$p_0=q_0+L_\beta^2, \qquad p_{-1}=q_{-1}+L^{-1}pL_\beta, \qquad p_{-2}=q_{-2}+p^2L^{-2}. \tag{2.2)1}$$

Moreover the reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij}=p^{-1}a^{ij}-s_0 b^i b^j-s_{-1}(b^i y^j+b^j y^i)-s_{-2} y^i y^j, \tag{2.3}$$

where we put

$$\begin{aligned} &b^i=a^{ij}b_j, \qquad s_0=(pp_0+(p_0p_{-2}-p_{-1}^2)\alpha^2)/\tau p, \\ &s_{-1}=(pp_{-1}-(p_0p_{-2}-p_{-1}^2)\beta)/\tau p, \\ &s_{-2}=(pp_{-2}+(p_0p_{-2}-p_{-1}^2)b^2)/\tau p, \quad b^2=a_{ij}b^i b^j, \\ &\tau=p(p+p_0b^2+p_{-1\beta})+(p_0p_{-2}-p_{-1}^2)(\alpha^2b^2-\beta^2). \end{aligned} \tag{2.3)1}$$

And the hv-torsion tensor $C_{ijk}=(\partial g_{ij}/\partial y^k)/2$ is given by

$$2p\,C_{ijk}=p_{-1}(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j)+r_{-1}m_i m_j m_k, \tag{2.4}$$

where we put

$$r_{-1}=pp_{0\beta}-3p_{-1}q_0, \qquad m_i=b_i-\alpha^{-2}\beta Y_i. \tag{2.4)1}$$

It is noted that the covariant vector m_i is a non-vanishing one, and is orthogonal to the supporting element y^i.

Remark. It is well-known [17] that a Finsler space $F^n(n\geq 5)$ with an (α,β)-metric is semi-C-reducible. The tensor C_{ijk} of such a space is written in the form

$$C_{ijk}=p(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)/(n+1)+qC_iC_jC_k/C^2,$$

where $C^2=C_iC^i$ and p is called the characteristic scalar of the F^n. Matsumoto and Numata have shown [15] that if the characteristic scalar p of the Finsler space with an (α,β)-metric is constant, then the space F^n is C-reducible.

The following identities hold:

$$\begin{aligned}&q_0\beta+q_{-1}\alpha^2=0, \quad q_{-1}\beta+q_{-2}\alpha^2=-p, \quad p_0\beta+p_{-1}\alpha^2=q,\\ &p_{-1}\beta+p_{-2}\alpha^2=0, \quad p^{-1}q_0-s_0u_0=0, \quad \alpha^2p_{-1\beta}+\beta p_{0\beta}=0,\\ &p_{-1}\alpha^2-p_0\beta^2+p_{-1}\alpha^3=0, \quad p_{-1}+p_{-1\beta}\beta+\alpha^2p_{-2\beta}=0,\\ &p_{0\alpha}\beta+p_{-1\alpha}\alpha^2+p_{-1}\alpha=0, \quad \beta p_{0\beta}+\alpha p_{0\alpha}=0,\end{aligned} \tag{2.5}$$

where $u_0=p+q_0b^2+q_{-1}\beta$ and $q=LL_\beta$. In virtue of (2.5), the scalars s_0, s_{-1} and τ in (2.3)1) are written in the form

$$\begin{aligned}&s_0=L^2q_0/\tau p\alpha^2, \quad s_{-1}=L^2p_{-1}/\tau p\alpha^2\\ &\tau=L^2\alpha^{-2}(p+\nu q_0).\end{aligned} \tag{2.6}$$

In order to prove the main results of this paper, we shall show three lemmas as follows. First

Lemma 1 ([18]). *The scalars h_0 and h_1 satisfying the equation*

$$h_0b_i+h_1Y_i=0$$

must be equal to zero.

Secondly, from (2.1), (2.2)1), (2.3)1), (2.4)1), (2.5) and lemma 1, we can show

Lemma 2. *The scalars r_{-1} and p_{-1} satisfy identically the following*

$$pp_{-1\beta}+\alpha^{-2}\beta r_{-1}+3p_{-1}pqL^{-2}-3p_{-1}{}^2=0. \tag{2.7}$$

Lemma 3 ([18]). *If a polynomial of degree 1 or 2 with respect to $\nu=b^2-\alpha^{-2}\beta^2$, whose coefficients are the function of α and β only, then all coefficients vanish, provided that b^2 be not constant.*

Definition. Let the F^n be a Finsler space with an (α,β)-metric on M^n. The Riemannian space $R^n=(M^n,\alpha)$ is called *the associated Riemannian space with the F^n*, and the Riemannian connection is called *the associated one.*

We denote by ∇_k the covariant differentiation by x^k with respect to the associated Riemannian connection, and $\{_j{}^i{}_k\}$ are the Christoffel's symbols of R^n. We put

$$(2.8)\quad \begin{aligned} &b_{jk}=\nabla_k b_j=\partial b_j/\partial x^k-b_r\{_j{}^r{}_k\},\\ &2E_{jk}=b_{jk}+b_{kj}, \qquad 2F_{jk}=b_{jk}-b_{kj}. \end{aligned}$$

For the sake of brevity, we denote by ∂_i and $\dot{\partial}_i$ partial differentiations with respect to x^i and y^i respectively. A direct calculation then leads us to

$$(2.9)\quad \begin{aligned} \gamma_i{}^k{}_j &= g^{kh}(\partial_j g_{ih}+\partial_i g_{hj}-\partial_h g_{ij})/2\\ &=\{_i{}^k{}_j\}+N^k E_{ij}+N_i F^k{}_j+N_j F^k{}_i+\{_0{}^s{}_j\}C_i{}^k{}_s+\{_0{}^s{}_i\}C_j{}^k{}_s\\ &\quad-\{_0{}^s{}_m\}g^{mk}C_{ijs}+b_{0j}B^k{}_i+b_{0i}B^k{}_j-b_{0m}g^{mk}B_{ij}, \end{aligned}$$

where we put

$$(2.10)\quad \begin{aligned} &N_k=p_0 b_k+p_{-1}Y_k, \qquad N^i=g^{im}N_m, \qquad F^k{}_i=g^{kr}F_{ri},\\ &B_{ik}=\{p_{-1}(a_{ik}-\alpha^{-2}Y_iY_k)+p_{0\beta}m_im_k\}/2, \quad B^k{}_i=g^{kj}B_{ij}. \end{aligned}$$

The suffix "0" means the contraction by y^i. For the symmetric tensor B_{ik} and the covariant vector N_k in (2.10), it follows from (2.2)1) and (2.5) that

$$(2.11)\quad B_{i0}=0, \qquad N_0=q, \qquad \dot{\partial}_i N_k=2B_{ik}.$$

Next, (2.9) easily yields

$$(2.12)\quad G^i:=(\gamma_j{}^i{}_k y^j y^k)/2=(\{_0{}^i{}_0\}+N^iE_{00}+2qF^i{}_0)/2.$$

Differentiating this by y^j, we obtain

$$(2.13)\quad \begin{aligned} G^i{}_j:=\dot{\partial}_j G^i &= \{_j{}^i{}_0\}+(B^i{}_j-N^mC_m{}^i{}_j)E_{00}+N^iE_{j0}\\ &\quad+q(F^i{}_j-2F^m{}_0C_m{}^i{}_j)+N_jF^i{}_0. \end{aligned}$$

Now we shall find the connection coefficients $F_j{}^i{}_k$ of the Cartan connection. It follows from (2.4), (2.8) and (2.13) that

$$(2.14)\quad \begin{aligned} F_j{}^i{}_k &:=\{_j{}^i{}_k\}+C_{jkr}G^r{}_m g^{im}-C_k{}^i{}_r G^r{}_j-C_j{}^i{}_r G^r{}_k\\ &=\{_j{}^i{}_k\}+N^iE_{jk}+F^i{}_kN_j+F^i{}_jN_k+B^i{}_jb_{0k}+B^i{}_kb_{0j}\\ &\quad-b_{0m}g^{mi}B_{jk}-C_j{}^i{}_mA^m{}_k-C_k{}^i{}_mA^m{}_j+C_{jkm}A^m{}_sg^{is}\\ &\quad+\lambda^s(C_j{}^i{}_mC_s{}^m{}_k+C_k{}^i{}_mC_s{}^m{}_j-C_k{}^m{}_jC_m{}^i{}_s), \end{aligned}$$

where we put

$$(2.15)\quad \begin{aligned} &A^m{}_k=B^m{}_kE_{00}+N^mE_{k0}+N_kF^m{}_0+qF^m{}_k,\\ &\lambda^s=N^sE_{00}+2qF^s{}_0. \end{aligned}$$

It follows from (2.11) that $A^m{}_0=\lambda^m$.

§ 3. The difference tensor $D_j{}^i{}_k$.

We introduce the quantities

$$D_j{}^i{}_k=F_j{}^i{}_k-\{{}_j{}^i{}_k\} \tag{3.1}$$

which are components of a tensor of (1,2)-type and symmetric in j, k. This tensor $D_j{}^i{}_k$ is called *the difference tensor* by Matsumoto ([9]). Then (2.14) gives

$$\begin{aligned} D_j{}^i{}_k=&N^iE_{jk}+F^i{}_kN_j+F^i{}_jN_k+B^i{}_jb_{0k}+B^i{}_kb_{0j}\\ &-b_{0m}g^{im}B_{jk}-C_j{}^i{}_mA^m{}_k-C_k{}^i{}_mA^m{}_j+C_{jkm}A^m{}_sg^{is}\\ &+\lambda^s(C_j{}^i{}_mC_s{}^m{}_k+C_k{}^i{}_mC_s{}^m{}_j-C_k{}^m{}_jC_m{}^i{}_s). \end{aligned} \tag{3.2}$$

Contracting this by y^k and then by y^j, we obtain

$$D_j{}^i{}_0=N^iE_{j0}+F^i{}_0N_j+B^i{}_jE_{00}+qF^i{}_j-C_m{}^i{}_j(N^mE_{00}+2qF^m{}_0), \tag{3.3}$$

$$D_0{}^i{}_0=N^iE_{00}+2qF^i{}_0, \tag{3.4}$$

where we used the relation (2.11). Therefore we see from (2.15) (3.3) and (3.4) that

$$D_0{}^i{}_0=\lambda^i,\qquad A^i{}_j=D_j{}^i{}_0+C_m{}^i{}_jD_0{}^m{}_0.$$

Further, on making use of these, $D_j{}^i{}_k$ is rewritten as

$$\begin{aligned} D_j{}^i{}_k=&N^iE_{jk}+F^i{}_kN_j+F^i{}_jN_k+B^i{}_jb_{0k}+B^i{}_kb_{0j}\\ &-b_{0m}g^{im}B_{jk}-C_j{}^i{}_mD_0{}^m{}_k-C_k{}^i{}_mD_j{}^m{}_0+C_{jkm}D_0{}^m{}_sg^{is}. \end{aligned} \tag{3.2$'$}$$

Contracting (3.4) by Y_i and m_i, in virtue of (2.3), we get respectively

$$Y_iD_0{}^i{}_0=\alpha^2L^{-2}qWH,\qquad m_iD_0{}^i{}_0=(q_0\nu E_{00}+2qF_{\beta0})W, \tag{3.5}$$

where we put

$$\nu=b^2-\alpha^{-2}\beta^2,\quad H=pE_{00}-2pF_{\beta0},\quad W=p^{-1}-s_0\nu. \tag{3.6}$$

These quantities in (3.5) and (3.6) are often used later on. A Finsler space is called *a Berwald space,* if the tensor $C_{ijk|l}$ vanishes identically (the coefficients $F_j{}^i{}_k$ are functions of position only). Here we shall show

Theorem 1. *The difference tensor $D_j{}^i{}_k$ vanishes if and only if the covariant vector b_i is parallel with respect to the associated Riemannian connection, and then the space F^n becomes a Berwald one.*

Proof. Suppose $b_{ij}=0$. It follows then from (2.8), (2.15) and (3.2) that $D_j{}^i{}_k=0$. Conversely, if $D_j{}^i{}_k=0$, from (3.4) we obtain

$$N^i E_{00}+2qF^i{}_0=0. \tag{3.7}$$

Multiplying (3.7) by l_i and summing over i, we have

$$E_{00}=0 \quad i.e., \qquad b_{ij}+b_{ji}=0. \tag{3.8}$$

Substituting from (3.8) into (3.7) and further contracting the result by g_{ij}, we get $F_{j0}=0$ that is

$$b_{ij}-b_{ji}=0. \tag{3.9}$$

Two equations (3.8) and (3.9) lead us to $b_{ij}=0$. Consequently the proof is complete.

Next, the h- and v-covariant derivatives $X_{i|j}, X_i|_j$ of a covariant vector field X_i with respect to the Cartan connection $C\Gamma$ are defined by

$$X_{i|j}=\partial_j X_i-(\dot{\partial}_r X_i)G^r{}_j-X_r F_j{}^r{}_i,$$
$$X_i|_j=\dot{\partial}_j X_i-X_r C_j{}^r{}_i.$$

For the covariant vector b_i, it follows from (2.8) and (3.1) that $b_{i|j}=b_{ij}-b_r D_j{}^r{}_i$, which gives

$$b_{i|0}=b_{i0}-b_r D_i{}^r{}_0. \tag{3.10}$$

Since $\alpha^2=a_{ij}(x)y^i y^j$ and $\beta=b_i y^i$, from (2.5), (3.1) and (3.10) we get

$$\alpha_{|h}=-\alpha^{-1}Y_m D_h{}^m{}_0, \qquad \beta_{|h}=b_{0h}-b_s D_h{}^s{}_0. \tag{3.11}$$

Further, for m_i, p, p_{-1} and r_{-1}, it follows from (3.10) and (3.11) that

$$\begin{aligned} &m_{i|h}=b_{ih}-m_r D_i{}^r{}_h-\alpha^{-2}Y_i Q_{0h}+\alpha^{-2}\beta(a_{is}-\alpha^{-2}Y_i Y_s)D_0{}^s{}_h,\\ &p_{|h}=p_{-1}Q_{0h}, \qquad p_{-1|h}=p_{-1\beta}Q_{0h}+\alpha^{-2}p_{-1}Y_s D_0{}^s{}_h,\\ &r_{-1|h}=r_{-1\beta}Q_{0h}+\alpha^{-2}r_{-1}Y_s D_0{}^s{}_h, \end{aligned} \tag{3.12}$$

where we put $Q_{0h}=b_{0h}-m_s D_0{}^s{}_h$. From (2.4), we obtain

$$\begin{aligned} 2pC_{ijk|l}=\;&h_{ij}A_{kl}+h_{jk}A_{il}+h_{ki}A_{jl}+A_{kl}'m_i m_j\\ &+A_{il}'m_j m_k+A_{jl}'m_k m_i, \end{aligned} \tag{3.13}$$

where we put

$$A_{kj}=p_{-1}m_{k|j}+(p_{-1|j}-p_{|j}p^{-1}p_{-1})m_k, \tag{3.14)1}$$

$$A_{kj}'=r_{-1}m_{k|j}+(r_{-1|j}-p_{|j}p^{-1}p_{-1})m_k/3. \tag{3.14)2}$$

It follows that $A_{0j}=A_{0j}'=0$. Here, from lemma 2 and (3.12), we get

$$p_{-1}r_{-1|j}-3r_{-1}p_{-1|j}+2r_{-1}p_{-1}p^{-1}p_{|j}=0. \tag{3.15}$$

Therefore, on making use of (3.15), A_{kj}' of (3.14)2) is rewritten as $A_{kj}'=r_{-1}(p_{-1})^{-1}A_{kj}$, so $C_{ijk|l}$ yields

(3.13)′ $$2pC_{ijk|l}=C_{ij}A_{kl}+C_{jk}A_{il}+C_{ki}A_{jl},$$

where the tensor C_{ij} is given by

(3.13)′1) $$p_{-1}C_{ij}=p_{-1}h_{ij}+r_{-1}m_i m_j.$$

Since $h_{ij}y^j=0$ and $m_i y^i=0$, we see $C_{i0}=0$.

Assume $C_{ijk|l}=0$, that is, the F^n be a Berwald space. Then (3.13)′ implies

(3.16) $$C_{ij}A_{kl}+C_{jk}A_{il}+C_{ki}A_{jl}=0.$$

Contracting (3.16) by g^{ij} and referring to (3.13)′1), we get

(3.17) $$\{(n+1)p_{-1}+r_{-1}m^2\}A_{kl}+2r_{-1}m_k A_{sl}m^s=0.$$

Further, contraction of this by m^k yields

(3.18) $$\{(n+1)p_{-1}+3r_{-1}m^2\}A_{sl}m^s=0.$$

which gives $A_{sl}m^s=0$, because of lemma 3 and $m^2=\nu W$. Thus from (3.17) we have $A_{kl}=0$. Conversely, if $A_{kl}=0$, we easily obtain $C_{ijk|l}=0$. Summarizing up the above, we have

Proposition 1. *A Finsler space with an (α,β)-metric is a Berwald space if and only if the tensor A_{kl} given by* (3.14)1) *vanishes.*

Here, we define the classification of Finsler spaces with an (α,β)-metric function $L(\alpha,\beta)$ as follows:

Definition. Let the F^n be a Finsler space with an (α,β)-metric function $L(\alpha,\beta)$.

(1) The F^n is called *of the first kind*, if the fundamental function $L(\alpha,\beta)$ is given by

(3.19)1) $$L(\alpha,\beta)=\alpha^t\beta^{1-t},$$

where $t(\neq 0,1)$ is any constant.

(2) The F^n is called *of the second kind*, if $L(\alpha,\beta)$ is given by

(3.19)2) $$L(\alpha,\beta)=(\alpha^s+a_1\alpha^{s-1}\beta+\cdots+a_k\alpha^{s-k}\beta^k+\cdots+\beta^s)^r,$$

where $rs=1$ and $a_1, a_2,\cdots=$const.

(3) The F^n is called *of the third kind*, if $L(\alpha,\beta)$ are given by the other forms of the exceptional case of the above (1), (2).

Examples. (1) The following well-known metric functions are examples of an (α,β)-metric of the first kind: (i) $t=2$: $L=\alpha^2/\beta$ (Kropina metric) (ii) $t=m+1$: $L=\alpha^{m+1}/\beta^m$ (generalized Kropina metric). (2) The Randers metric $L=\alpha+\beta$ is an (α,β)-metric of the second kind. (3) The following metric is of the third kind : $L=\alpha^{\psi}\beta^{\zeta}(\alpha^{s}+a_{1}\alpha^{s-1}\beta+\cdots+a_{k}\alpha^{s-k}\beta^{k}+\cdots+\alpha^{s})^{r}$, where $\psi\zeta sr=1$.

Again, we continue the discussion of a Berwald space. Kikuchi has shown [7] that the Randers space is a Berwald space iff $\nabla_{k}b_{i}=0$, and the Kropina space (resp. The generalized Kropina space) is a Berwald space iff there exists the covariant vector $\gamma_{k}(x)$ such that the following equation holds good;

$$b_{ik}=(\gamma_{k}b_{i}-\gamma_{i}b_{k}+\gamma^{r}b_{r}a_{ik})/2,$$
$$(\text{resp. } b_{ik}=(\gamma_{k}b_{i}/m-\gamma_{i}b_{k}+b_{r}\gamma^{r}a_{ik})/2,).$$

On the other hand, Matsumoto and Numata have shown a similar result in [14] that the Finsler space with a decomposition of a cubic metric $L=(\alpha^{2}\beta)^{1/3}$ is a Berwald space there exists the covariant vector $\gamma_{k}(x)$ such that $a_{ij|k}=\gamma_{k}(x)a_{ij}$ and $b_{i|k}=-\gamma_{k}b_{i}$. Assume that the F^{n} is a Berwald space. First, we consider a Finsler space with an (α,β)-metric of the first kind. Differentiating (3.19)1) h-covariantly, we have

$$(3.20)\qquad \beta\omega\alpha^{2}{}_{|i}-\alpha^{2}\beta_{|i}=0,\qquad \omega=t/2(t-1).$$

Since the Cartan connection coefficients $F_{j}{}^{i}{}_{k}$ are independent of y^{i} and the quadratic form $a_{jk}y^{j}y^{k}$ is positive definite, we have

$$(3.21)\qquad a_{jk|i}=\gamma_{i}a_{jk},\qquad b_{j|i}=\omega\gamma_{i}b_{j}.$$

Conversely, if there exists a non-zero vector $\gamma_{i}(x)$ such that the equation (3.21) holds good, it then follows that

$$(3.22)\qquad \alpha^{2}{}_{|i}=\gamma_{i}\alpha^{2},\qquad \beta_{|i}=\omega\gamma_{i}\beta.$$

Here, referring to $L_{|i}=0$, we obtain

$$(3.23)\qquad p\alpha^{2}{}_{|i}+2q\beta_{|i}=0.$$

The equations (3.22) and (3.23) imply $p\alpha^{2}+2q\beta=0$. Since $L^{2}=\alpha^{2}p+\beta q$ and $q=LL_{\beta}$, we have $L=(1-2\omega)\beta\partial L/\partial\beta$. Integrating this and paying attention to the homogeneity of L, we easily get $L=c_{1}\alpha^{t}\beta^{1-t}$, where c_{1} is an arbitrary constant. It will be easy, however, to show that the above L is reduced to $L=\alpha'^{t}\beta^{1-t}$ by a suitable change of α; that is, the space F^{n} is of the first kind. Moreover, (3.21) implies

$$(3.24)1)\qquad F_{j}{}^{i}{}_{k}=\{{}_{j}{}^{i}{}_{k}\}-(\gamma_{k}\delta^{i}{}_{j}+\gamma_{j}\delta^{i}{}_{k}-\gamma^{i}a_{jk})/2,$$

$$(3.24)2)\qquad b_{ik}=\{(2\omega-1)\gamma_{k}b_{i}-\gamma_{i}b_{k}+\gamma^{m}b_{m}a_{ik}\}/2,\quad (\gamma^{i}=a^{im}\gamma_{m}).$$

We see that $F_{j}{}^{i}{}_{k}$ has linear connection and symmetry. Owing to (3.24)1), F^{n} is a Berwald space. Summarizing up the above we have

Theorem 2. (1) *If there exists the covariant vector* $\gamma_i(x)$ *such that* $a_{jk|i}=\gamma_i a_{jk}$ *and* $b_{j|i}=\omega\gamma_i b_j$, *the Finsler space with an* (α,β)*-metric is of the first kind.* (2) *Let the Finsler space* F^n *be of the first kind. Then the* F^n *is a Berwald space if and only if there exists the covariant vector* $\gamma_k(x)$ *such that* (3.24)2) *holds good.*

Next, we treat the Finsler space F^n with an (α,β)-metric of the second kind. Assume that F^n is a Berwald space. By means of (3.19)2) and $L_{|i}=0$, we have

$$\begin{aligned}(3.25)\qquad &\alpha^2{}_{|i}(s\alpha^{s-1}+a_1(s-1)\alpha^{s-2}\beta+\cdots+a_k(s-k)\alpha^{s-k-1}\beta^k+a_{s-1}\beta^{s-1})\\ &+2\beta_{|i}(a_1\alpha^s+\cdots+a_k k\alpha^{s-k+1}\beta^{k-1}+\cdots+s\alpha\beta^{s-1})=0.\end{aligned}$$

As for the Finsler space F^n of the first kind, (3.25) gives

$$(3.26)\qquad (1)\quad a_{ij|k}=0, \qquad (2)\quad b_{i|k}=0,$$

so (3.26)1) implies

$$(3.27)\qquad F_j{}^i{}_k=\{{}_j{}^i{}_k\}.$$

Therefore, in virtue of (3.26) and (3.27), we get

$$(3.28)\qquad b_{ik}=0.$$

Conversely, assume that the equation (3.26) holds good. For a Finsler space, the following equation is known

$$(3.29)\qquad L^2{}_{|i}=\partial_i L^2-2y_r G^r{}_i=0.$$

And it follows that

$$(3.30)\qquad G^r{}_i=(\dot\partial_i\gamma_j{}^r{}_k)y^j y^k/2+\gamma_i{}^r{}_0.$$

Since (2.5) and (2.11), we have

$$(3.31)\qquad \dot\partial_r p=p_{-1}m_r, \qquad \dot\partial_r q=N_r.$$

Thus the h-covariant differentiation of the scalars α^2, β, p and q lead us to

$$(3.32)\qquad \begin{aligned}&\alpha^2{}_{|i}=\partial_i\alpha^2-2Y_m G^m{}_i, &&\beta_{|i}=\partial_i\beta-b_r G^r{}_i,\\ &p_{|i}=\partial_i p-p_{-1}m_r G^r{}_i, &&q_{|i}=\partial_i q-N_r G^r{}_i.\end{aligned}$$

On the other hand, from $L^2=\alpha^2 p+\beta q$, we have the following differential equation for a Finsler space with an (α,β)-metric

$$(3.33)\qquad \partial_i L^2-p\partial_i\alpha^2-q\partial_i\beta-\alpha^2\partial_i p-\beta\partial_i q=0.$$

Moreover, referring to $L^2{}_{|i}=0$, we obtain

$$(3.34)\qquad p\alpha^2{}_{|i}+\alpha^2 p_{|i}+q\beta_{|i}+\beta q_{|i}=0.$$

The equation (3.23) and (3.34) give

(3.35) $\quad \alpha^2 p_{|i} + \beta q_{|i} - q\beta_{|i} = 0.$

which is equivalent to

(3.36) $\quad \alpha^2 \partial_i p + \beta \partial_i q - q \partial_i \beta = 0,$

because of (2.5) and (3.32). Substituting from (3.36) into (3.33), we get

(3.33)′ $\quad \partial_i L^2 - p \partial_i \alpha^2 - 2 q \partial_i \beta = 0.$

On the other hand, from the definition of $\gamma_j{}^i{}_k$ and $\{{}_j{}^i{}_k\}$, referring to $L^2 = y^i y_i$ and $Y_m = a_{mi} y^i$, (3.27) gives

(3.37) $\quad g^{hm}(2(\partial_r Y_m) y^r - \partial_m L^2) = a^{hm}(2 y^r \partial_r Y_m - \partial_m \alpha^2).$

The contraction of this by g_{hi} implies $2\Delta y_i - \partial_i L^2 - p(2\Delta Y_i - \partial_i \alpha^2) - p_{-1} \Delta \alpha^2 m_i - (2\Delta Y_m - \partial_m \alpha^2) b^m N_i = 0$, where we put $\Delta = y^r \partial_r$. Form (3.3)′ this is rewritten in the form

(3.38) $\quad 2\Delta y_i - 2p(\Delta Y_i) - p_{-1}(\Delta \alpha^2) m_i - 2 q \partial_i \beta - 2(\Delta Y_m - \partial_m \alpha^2) b^m N_i = 0.$

Conversely, if the differential equation (3.38) holds good, then we immediately get the equation (3.37), so that it follows from (3.1) that $\gamma_0{}^i{}_0 = \{{}_0{}^i{}_0\}$, which implies $D_0{}^i{}_0 = 0$. Thus from Theorem 1 we get $b_{ij} = 0$ and $a_{ij|k} = 0$. Consequently, we have

Theorem 3. *The Finsler space F^n with an (α, β)-metric of the second kind is a Berwald space if and only if $\nabla_j b_i = 0$ satisfies or the partial differential equation* (3.38) *holds good.*

We have not yet studied the Finsler space with an (α, β)-metric of the third kind.

§ 4. The hv-curvature tensor P_{hijk}.

The hv-curvature tensor is defined by

$$P_{hijk} = C_{ijk|h} - C_{hjk|i} + C_{hjr} C_i{}^r{}_{k|0} - C_{ijr} C_h{}^r{}_{k|0}.$$

Owing to (3.13)′, P_{hijk} is written in the form

(4.1) $$P_{hijk} = \mathfrak{A}_{(hi)} \{ h_{hj} M_{ki} + h_{ki} B_{hj} + h_{kj} M_{hi} + m_h m_j D_{jh} + m_k m_j H_{ih} \},$$

where we put

(4.2)
$$2pM_{ki} = -A_{ki} + ((p_{-1} + r_{-1} m^2) m_k A_i + m^r A_r A_k A_i)/2p,$$
$$2pB_{ki} = A_{ki} + p_{-1}(m_i A_k + h_{ki} A_r m^r)/2p,$$
$$2pD_{ki} = r_{-1}(p_{-1})^{-1}[-A_{ki} + (2p_{-1} + r_{-1} m^2) m_k A_i + p_{-1} A_r m^r h_{ki}/2p],$$
$$2pH_{ki} = r_{-1}(p_{-1})^{-1}(A_{ki} + p p_{-1} h_{ki} A_r m^r/2p),$$

and $\mathfrak{A}_{(hi)}$ means the interchange of indices h, i and subtraction. Here, for the tensors in (4.2), it follows that

(4.3) $$2pM_{k0}=-2pB_{k0}=-A_k, \qquad 2pp_{-1}D_{k0}=-2pp_{-1}H_{k0}=r_{-1}A_k,$$

where we put $A_{k0}=A_k$.

Definition. A Finsler space is called a *Landsberg space* if the (v)hv-torsion tensor P_{ijk} $(=C_{ijk|0})$ vanishes.

It is noted that the condition $P_{ijk}=0$ is equivalent to $P_{hijk}=0$. In this section we shall find a condition for a Finsler space with an (α,β)-metric to be a Landsberg space. The contraction of (4.1) by y^h, in virtue of (4.3), gives

(4.4) $$2pP_{ijk}=C_{ij}A_k+C_{jk}A_i+C_{ki}A_j,$$

where the tensor C_{ij} is of (3.13)′1). Further, contracting (4.4) by g^{jk} and then by m^i, we get

(4.5) (1) $$2pp_{-1}P_{ijk}g^{jk}=((n+1)p_{-1}+r_{-1}m^2)A_i+2r_{-1}A_m m_i,$$
(2) $$2pp_{-1}P_{kij}g^{jk}m^i=((n+1)p_{-1}+3r_{-1}m^2)A_m,$$

where we put $A_m=A_k m^k$ and m^2 is equal to $\nu(p^{-1}-s_0\nu)$. Suppose that $P_{ijk}=0$. Then from (4.5)2) and lemma 3 we have $A_m=0$, so (4.5)1) gives $A_k=0$. Thus we have

Proposition 2. *A Finsler space with an (α,β)-metric is a Landsberg space if and only if the vector A_i $(=A_{i0})$ in (4.3) vanishes.*

Further we continue the discussion of the conditions of a Landsberg space. From (3.14)1) and $A_i=0$, referring to $p_{-1}=0$, we have

(4.6) $$m_{k|0}=-m_k(p_{-1|0}-p_{|0}p^{-1}p_{-1})/p_{-1}.$$

Here, the contraction of (3.12) by y^h gives

(4.7) $$m_{i|0}=b_{i0}-m_r D_i{}^r{}_0-\alpha^{-2}WH\,Y_i+\alpha^{-2}\beta(a_{is}-\alpha^{-2}Y_iY_s)D_0{}^s{}_0,$$
$$p_{|0}=p_{-1}WH, \qquad p_{-1|0}=(p_{-1\beta}+p_{-1}L^{-2}q)WH.$$

In particular, we assume that $H=0$. It then follows from (4.7) that $p_{|0}=p_{-1|0}=r_{-1|0}=0$ and $Y_sD_0{}^s{}_0=Q_{00}=WH=0$. Therefore, from these results and (4.7) we have

(4.8) $$m_{k|0}=E_{k0}+F_{k0}-m_sD_k{}^s{}_0+\alpha^{-2}\beta a_{ks}D_0{}^s{}_0=0.$$

Contracting this by b^k and referring to (3.3) we obtain

(4.9) $$pWE_{\beta0}+\{1+\alpha^{-2}\beta qs_0-p_0\nu W+p_{-1}p^{-1}q(m^2+2s_0\nu^2$$
$$+s_0\nu(p+q_0\nu)m^2-2p^{-1}q_0\nu m^2)\}F_{\beta0}=0.$$

Differentiating $H=pE_{00}-2qF_{\beta0}$ by y^k and contracting the result by b^k, in virtue of $Y_kb^k=\beta$, $b_kb^k=b^2$ and (2.5), we obtain

(4.10) $\quad pE_{\beta 0}=\{\alpha^{-2}\beta q+\nu(q^2L^{-2}+q_0-p_{-1}p^{-1}q)\}F_{\beta 0}.$

Since $\tau W=\alpha^{-2}L^{-2}$ and $\tau s_0=p^{-1}q_0\alpha^{-2}L^2$, substituting from (4.10) into (4.9) and multiplying the results by $\tau\alpha^2L^{-2}$, we get $F_{\beta 0}=0$, i.e. $F_{\beta i}=0$, so that which leads us to $E_{00}=0$, i.e. $E_{ij}=0$, because of $H=0$. Finally, substituting from $F_{\beta k}=E_{00}=0$ into (3.3), (3.4) and (4.8), we have $F_{k0}=0$, i.e. $F_{ki}=0$, which implies $b_{ij}=0$. Consequently the Finsler space with an (α,β)-metric is a Berwald space, because of Theorem 1. Again we return to the general equation (4.6). The contraction $m_{k|0}$ in (4.7) by b^k yields

(4.11) $\quad m_{k|0}b^k=E_{\beta 0}+F_{\beta 0}-m_sD_k{}^s{}_0b^k-\alpha^{-2}\beta WH+\alpha^{-2}\beta(s_0\nu H+2qp^{-1}F_{\beta 0}).$

Substituting from (4.6) into (4.7) and contracting it by b^k, by a somewhat complicated computation, we get

(4.12) $\quad p^{-1}(q_0f_1-q_0^2p^{-1}f_2-p_{-1}r_{-1}H/2)\nu^2+f_1\nu+f_2=0,$

where we put

$$\begin{aligned} f_1&=p_{-1}\{pq_0E_{\beta 0}+(2pq_0-pp_0+qq_0\alpha^{-2}\beta+2p_{-1}qF_{\beta 0}\\ &\quad -p_{-1}pE_{00}/2-\alpha^{-2}\beta(r_{-1}+2p_{-1}q_0)H,\\ f_2&=pp_{-1}(pE_{\beta 0}+pF_{\beta 0}-\alpha^{-2}\beta H). \end{aligned}\tag{4.13}$$

Here, if E_{00} and $F_{\beta 0}$ are functions of α and β only, then, by means of lemma 3, the coefficients of a polynomial with respect to ν in (4.12) vanish so we have $H=0$ or $r_{-1}=0$, because of $f_1=f_2=0$. On the other hand, Matsumoto has shown [8] that a C-reducible Landsberg space is a Berwald space. Our Finsler space with an (α,β)-metric is not necessarily C-reducible in general, but if $r_{-1}=0$, then the space F^n is C-reducible [7]. Summarizing up the above, we can conclude

Theorem 4. *Let the Finsler space F^n with an (α,β)-metric be a Landsberg space. Then, F^n becomes a Berwald space if and only if one of the following conditions hold good:*
(1) *The scalar H given by* (3.6) *vanishes*
(2) *The scalar r_{-1} vanishes*
(3) *The scalars E_{00} and $F_{\beta 0}$ are functions of α and β only.*

Next, Matsumoto ([10]) introduced the notion of P-reducibility of Finsler spaces : A Finsler space is called *P-reducible,* if the torsion fensor P_{ijk} is of the form

$$P_{ijk}=(h_{ij}P_k+h_{jk}P_i+h_{ki}P_j)/(n+1).$$

It is well-known that any C-reducible Finsler space is P-reducible. Referring to (3.13)′ and (4.5)1), P_{ijk} in (4.4) is rewritten as

$$\begin{aligned} \phi P_{ijk}=\mathfrak{S}_{(ijk)}\{&r_{-1}P_km_im_j+r_{-1}p^{-1}A_mh_{ij}m_k+p_{-1}h_{ij}P_k\\ &-2(p_{-1}p)^{-1}r_{-1}2A_mm_im_jm_k\}, \end{aligned}\tag{4.14}$$

where $\phi=(n+1)p_{-1}+r_{-1}m^2$ and $\mathfrak{S}_{(ijk)}$ means the cyclic permutation of indices i, j, k and summation. We assume that the Finsler space with an (α,β)-metric is P-reducible. Then the equation (4.14) gives

$$(4.15)\qquad \mathfrak{S}_{(ijk)}\{(p_{-1}-\phi/(n+1))h_{ij}P_k+r_{-1}P_k m_i m_j-r_{-1}p^{-1}A_m h_{ij}m_k -2(p_{-1}p)^{-1}r_{-1}^2 A_m m_i m_j m_k\}=0.$$

Contracting (4.15) by $m^i m^j m^k$ and referring to $m^2\neq 0$, we get $r_{-1}=0$, so the F^n is C-reducible. Therefore we have

Proposition 3. *A Finsler space F^n with an (α,β)-metric is P-reducible if and only if the space F^n is C-reducible.*

§ 5. The h-curvature tensor R_{hijk}.

We shall express the h-curvature tensor R_{hijk} in a concrete form. The tensor $R_j{}^i{}_{hk}$ is defined by

$$R_j{}^i{}_{hk}=\mathfrak{A}_{(hk)}\{\partial_k F_j{}^i{}_h-G^r{}_k\partial_r F_j{}^i{}_h+C_j{}^i{}_r(\partial_k G^r{}_h-G^l{}_k G_l{}^r{}_h) +F_r{}^i{}_k F_j{}^r{}_h\}.$$

On making use of the difference tensor $D_j{}^i{}_k$, $R_j{}^i{}_{hk}$ is written as

$$(5.1)\qquad R_j{}^i{}_{hk}=\bar{R}_j{}^i{}_{hk}+C_j{}^i{}_m\bar{R}_0{}^m{}_{hk}+\mathfrak{A}_{(hk)}\{D_j{}^i{}_{h|k}+D_r{}^i{}_h D_j{}^r{}_k +C_j{}^i{}_m(D_0{}^m{}_{h|k}+D_s{}^m{}_h D_0{}^s{}_k)\},$$

where $\bar{R}_j{}^i{}_{hk}$ is the Riemannian curvature tensor of the associated Riemannian space R^n. The contraction of (5.1) by y^j yields

$$(5.2)\qquad R_0{}^i{}_{hk}=\bar{R}_0{}^i{}_{hk}+D_0{}^i{}_{h|k}-D_0{}^i{}_{k|h}+D_r{}^i{}_h D_0{}^r{}_k-D_r{}^i{}_k D_0{}^r{}_h.$$

In the similar way as in the proof of Proposition 3 of the paper [22],we can show the following

Theorem 5. *The h-curvature tensor $R_j{}^i{}_{hk}$ of the Finsler space with an (α,β)-metric is written as*

$$(5.3)\qquad \bar{R}_j{}^i{}_{hk}=D_j{}^i{}_{k|h}-D_j{}^i{}_{h|k}+D_r{}^i{}_k D_j{}^r{}_h-D_r{}^i{}_h D_j{}^r{}_k.$$

It is shown that the h-curvature tensor $H_j{}^i{}_{hk}$ of Berwald connection $B\Gamma$ is written in the form

$$(5.4)\qquad H_{hijk}=K_{hijk}+\mathfrak{A}_{(ik)}\{C_{hmi|0}C_j{}^m{}_{k|0}+C_{hij|0|k}\},$$

where we put $K_{hijk}=R_{hijk}-C_{hjr}R_0{}^r{}_{ik}$. Therefore, from (5.1) and (5.4), we have

$$(5.5)\qquad H_j{}^i{}_{hk}=R_j{}^i{}_{hk}+\mathfrak{A}_{(hk)}\{D_j{}^i{}_{h|k}+D_r{}^i{}_h D_j{}^r{}_k+C_r{}^i{}_{h|0}C_j{}^r{}_{k|0}+C_j{}^i{}_{h|0|k}\}.$$

On the other hand, the relation between the h-curvature tensor R_{hijk} of Cartan connection $C\Gamma$ and the h-curvature tensor H_{hijk} of Berwald connection $B\Gamma$ is given by

$$R_{hijk}=H_{hijk}+C_{hijr}R^{r}{}_{jk}-P_{hij|k}+P_{hik|j}-Q_{hijk}, \tag{5.6}$$

where we put

$$Q_{hijk}=P_{irh}P_{j}{}^{r}{}_{k}-P_{irk}P_{h}{}^{r}{}_{j}. \tag{5.7}$$

Suppose $R_{hijk}=0$. From (5.3), (5.5) and (5.6), we have

$$Q_{hijk}=P_{irj}P_{h}{}^{r}{}_{k}-P_{irk}P_{h}{}^{r}{}_{j}=0. \tag{5.8}$$

Referring to $A_r h^r{}_k=A_k$, (4.3) and (5.8) give

$$\begin{aligned}\mathfrak{A}_{(jk)}\{&C_{hr}C^{r}{}_{j}A_kA_i+C_{hr}C_{ij}A^rA_k+C_{rk}C^{r}{}_{i}A_jA_h+C_{kr}C_{ij}A^rA_h\\&+C_{kh}C_{ir}A_jA^r+A_{kh}C_{jr}A_iA^r+C_{kh}C_{ij}A^2\}=0,\end{aligned} \tag{5.9}$$

where we put $A^2=A_iA_jg^{ij}$ and $C_j{}^r=C_{jm}g^{rm}$. The contraction of (5.9) by g^{hj} yields

$$\begin{aligned}&C_{sr}C^{rs}A_kA_i-2C_{hr}C_{ik}A^rA^h+2C_{rk}C^{r}{}_{i}A^2+2C_{rk}A^rC_{si}A^s\\&-C_s{}^s(A_kA_rC^r{}_i+A_iA_rC^r{}_k)-C_s{}^sC_{ik}A^2=0,\end{aligned} \tag{5.10}$$

where $C^{rs}=g^{rm}C_m{}^s$ and $C_s{}^s=g^{rs}C_{sr}$. Since $p_{-1}C_{ij}=p_{-1}h_{ij}+r_{-1}m_im_j$, (5.10) is rewritten in the form

$$a_1h_{ki}+a_2(m_iA_k+m_kA_i)+a_3A_kA_i+a_4m_im_k=0, \tag{5.10$'$}$$

where we put

$$\begin{aligned}&a_1=-p_{-1}\{(n-1)p_{-1}A^2+r_{-1}m^2A^2+2r_{-1}A_m{}^2\},\\&a_2=-r_{-1}A_m\{(n-3)p_{-1}+r_{-1}m^2\},\\&a_3=-(n-3)p_{-1}{}^2+r_{-1}{}^2m^4,\\&a_4=-r_{-1}A^2((n-3)p_{-1}-r_{-1}m^2),\quad A_m=A_km^k.\end{aligned} \tag{5.11}$$

Further, contracting (5.10)′ by g^{ik}, we have

$$(n-1)a_1+2A_ma_2+A^2a_3+m^2a_4=0. \tag{5.12}$$

On the other hand, the contraction (5.10)′ by m^i and then by m^k, A^k gives

$$a_1m_k+(a_2m^2+a_3A_m)A_k+(a_2A_m+a_4m^2)m_k=0, \tag{5.13}$$

$$a_1m^2+2a_2m^2A_m+a_3A_m{}^2+a_4(m^2)^2=0, \tag{5.14}$$

$$a_1A_m+a_2(A_m{}^2+A^2m^2)+a_3A_mA^2+a_4m^2A_m=0. \tag{5.15}$$

Moreover, contracting (5.10)′ by A^iA^k, we obtain

$$a_1A^2+2a_2A_mA^2+a_3A^4+a_4A_m{}^2=0. \tag{5.16}$$

Therefore, from (5.12), (5.14), (5.15) and (5.16) we get

(5.17) $$(A_m^2-m^2A^2)(A^2a_3-m^2a_4)=0,$$
$$(n-2)a_1m^2-(A_m^2-A^2m^2)a_3=0.$$

Here, if $A^2a_3-m^2a_4=0$, then (5.11) yields $A^2(r_{-1}m^2-p_{-1})=0(n\geqq 3)$, thus, which implies $A^2=0$, because of lemma 3.

On the other hand, if $A_m^2-m^2A^2=0$, the second equation of (5.17) gives $a_1=0$, so (5.11) leads us to $A^2=0$, that is, $A_k=0$, provided that g_{ij} be a positive definite. Consequently from Proposition 2, the F^n is a Landsberg space. Summarizing up the above, we can state

Theorem 6. *Let the h-curvature tensor R_{hijk} of the Finsler space with an (α,β)-metric be vanishing. Then the F^n becoms a Landsberg space.*

Corollary 6.1. *Let the h-curvature tensor R_{hijk} of the Finsler space F^n $(n\geqq 4)$ with an (α,β)-metric be vanishing. Then the F^n becomes a locally Minkowski space if one of the following conditions holds good:*

(1) *The scalar H given by* (3.6) *vanishes*

(2) *The scalar r_{-1} in* (2.4)1) *vanishes*

(3) *The scalars E_{00} and $F_{\beta 0}$ are functions of α and β only.*

In [13], we showd that a C-reducible Finsler space is locally Minkowski if and only if the h-curvature tensor R_{hijk} vanishes under the assumption that the Finsler metric is a positive definite. On the other hand, we treated a Berwald space in § 3, and if $b_{ij}=0$ and $\bar{R}_{hijk}=0$, from (5.1) we get $R_{hijk}=0$, so the F^n is locally Minkowski. Kikuchi has shown [7] that if the Kropina space (resp. the generalized Kropina space) is locally Minkowski iff the following equation

(5.18) $$C_j{}^i{}_{kh}:=\bar{R}_j{}^i{}_{kh}-2\mathfrak{A}_{(kh)}\{\delta^i{}_kL_{hj}+L^i{}_ka_{hj}\}/(n-2)=0$$

hold good, where $L_{hk}=-\bar{R}_{hk}+\bar{R}a_{hk}/2(n-1)$, $\bar{R}_{hk}=\bar{R}_h{}^i{}_{ki}$, $\bar{R}=\bar{R}_{hk}a^{hk}$, $L^i{}_k=a^{ij}L_{jk}$ and there exists the scalar $\rho(x)$ such that

(5.19) $$\nabla_i\rho_j-\rho_i\rho_j+\rho_m\rho^ma_{ij}/2=-L_{ij}/(n-2),$$
$$\nabla_kb_i=(\rho_ib_k-\rho_kb_i-\rho^rb_ra_{ik})\ \text{(resp. } \nabla_kb_i=\rho b_k-\rho_kb_i/m-b_r\rho^ra_{ik}).$$

In the similar way as in the proof of the above results, we can show

Theorem 7. (1) *The Finsler space F^n with an (α,β)-metric of the first kind is locally Minkowski if and only if* (5.18) *holds good and there exsits the scalar $\rho(x)$ such that* (5.19) *and*

$$\nabla_kb_i=-((1/t-1)\rho_kb_i-\rho_ib_k+\rho^rb_ra_{ik})$$

are satisfied.

(2) *The Finsler space F^n with an (α, β)-metric of the second kind is locally Minkowski if and only if $\nabla_i b_j = 0$ and $\bar{R}_{hijk} = 0$.*

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