Totally Contact Umbilical Submanifolds in Sasakian Manifolds

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佐々木多様体の全触臍的部分多様体

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Abstract

In our previous paper [4], we investigated the totally umbilical submanifolds in Sasakian manifolds.

In this paper, we shall examine the totally contact umbilical submanifolds in Sasakian manifolds.

§ 1. Statement of results.

In this paper, we shall prove the following theorems:

THEOREM 1. *Let M be a totally contact umbilical contact CR-submanifold in a Sasakian manifold $\tilde{M}$. If M is not anti-invariant and* $\dim M \geq 6$, *then the mean curvature vector μ is parallel and the anti-invariant distribution $D^{\perp}$ is equal to $\langle \tilde{\phi}\mu \rangle$.*

THEOREM 2. *Let M be a totally contact umbilical submanifold with non-zero parallel mean curvature vector μ in a Sasakian manifold $\tilde{M}$. If $\tilde{\phi}\mu$ is tangent to M, then M is a contact CR-submanifold with anti-invariant distribution $D^{\perp}=\langle \tilde{\phi}\mu \rangle$. Furthermore, M admits a homothetic second Sasakian structure.*

The following corollary is a direct consequence of Theorem 1 and 2.

COROLLARY. *Let M be an $n(\geq 6)$-dimensional totally contact umbilical submanifold with*

non-zero mean curvature vector μ in a Sasakian manifold $\tilde{M}$, and assume that M is not anti-invariant. Then M is a contact CR-submanifold if and only if the mean curvature vector μ is parallel and $\tilde{\phi}\mu$ is tangent to M. In this case M has an anti-invariant distribution $D^{\perp}=\langle\tilde{\phi}\mu\rangle$ and admits a homothetic second Sasakian structure.

§ 2. Preliminaries.

(1) Sasakian manifolds and second Sasakian manifolds.

Let M be a $(2m+1)$-dimensional manifold, ϕ a tensor field of type $(1,1)$, ξ a vector field and η a 1-form on M. If $(\phi,\ \xi,\ \eta)$ satisfies the conditions

(2.1) $\eta(\xi)=1$

and

(2.2) $\phi^2X=-X+\eta(X)\xi$

for any vector field X on M, then M is said to admit an *almost contact structure* (ϕ,ξ,η) and M is called an *almost contact manifold*. For the almost contact structure (ϕ,ξ,η), it follows that

(2.3) $\phi\xi=0,\quad \eta(\phi X)=0$ and rank $\phi=2m$.

Suppose that, on an almost contact manifold M, there is given a Riemannian metric $\langle\ ,\ \rangle$ satisfying

(2.4) $\langle\phi X,Y\rangle+\langle X,\phi Y\rangle=0$ and $\eta(X)=\langle X,\xi\rangle$.

Then the almost contact structure is said to be *metric* and M is called an *almost contact Riemannian manifold.*

If an almost contact metric structure satisfies

(2.5) $2c\langle\phi X,Y\rangle=d\eta(X,Y)$

for some non-zero constant c, then the almost contact metric structure is called a *homothetic contact metric structure*. M with such a structure is called a *homothetic contact Riemannian manifold*. When $c=1$ in (2.5), the structure is a *contact metric structure* and M with this structure is a *contact Riemannian manifold*.

If an almost contact structure (ϕ,ξ,η) satisfies

(2.6) $\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]+d\eta(X,Y)\xi=0$,

then the structure is said to be *normal*.

If a contact metric structure (resp., homothetic contact metric structure) is normal, the structure is said to be a *Sasakian structure* (resp., *homothetic Sasakian structure*) and M is called a *Sasakian manifold* (resp., *homothetic Sasakian manifold*).

It is well-known that the following identities hold in a homothetic Sasakian manifold:

(2.7) $\nabla_X\xi=c\phi X$.

(2.8) $(\nabla_X\phi)Y=c(\eta(Y)X-\langle X,Y\rangle\xi)$.

An almost contact metric structure $(\phi,\xi,\eta,\langle\ ,\ \rangle)$ on M is a homothetic Sasakian structure if and only if (2.8) holds.

The Riemannian curvature tensor R of a Sasakian manifold M satisfies

$$R(\phi X,\phi Y)Z=R(X,Y)Z-\langle Y,Z\rangle X+\langle Z,X\rangle Y-\langle Y,\phi Z\rangle\phi X+\langle\phi Z,X\rangle\phi Y,$$
$$R(X,\phi Y)Z=-R(\phi X,Y)Z-\langle Y,\phi Z\rangle X+\langle\phi Z,X\rangle Y+\langle Y,Z\rangle\phi X-\langle Z,X\rangle\phi Y,$$
(2.9) $$R(X,Y)\xi=\eta(Y)X-\eta(X)Y$$
and
$$R(\xi,X)Y=\langle X,Y\rangle\xi-\eta(Y)X.$$

Next, let M be a $(2m+2)$-dimensional manifold. If there exist five tensor fields ϕ, ξ_1, ξ_2, η_1 and η_2 on M of types (1,1), (1,0), (1,0), (0,1) and (0,1) respectively which satisfy

(2.10) $$\eta_a(\xi_b)=\delta_{ab},\ \phi(\xi_a)=0 \qquad (a,b=1,2),$$
$$\phi^2X=-X+\eta_1(X)\xi_1+\eta_2(X)\xi_2$$

for any vector field X on M, then M is said to admit an *almost 2-contact structure* $(\phi,\xi_1,\xi_2,\eta_1,\eta_2)$ and M is called an *almost 2-contact manifold*.

For the almost 2-contact structure $(\phi,\xi_1,\xi_2,\eta_1,\eta_2)$, it follows that

(2.11) $$\eta_a(\phi X)=0 \qquad (a=1,2), \qquad \text{rank } \phi=2m.$$

Suppose that, on an almost 2-contact manifold M, there exists a Riemannian metric $\langle\ ,\ \rangle$ satisfying

(2.12) $$\eta_a(X)=\langle X,\xi_a\rangle \qquad (a=1,2),$$
$$\langle\phi X,Y\rangle+\langle X,\phi Y\rangle=0.$$

Then the almost 2-contact structure is said to be *metric* and M is called an *almost 2-contact Riemannian manifold*.

If an almost 2-contact metric structure satisfies

(2.13) $$2c_a\langle\phi X,Y\rangle=d\eta_a(X,Y) \qquad (a=1,2)$$

for non-zero constants c_1 and c_2, the structure is called a *homothetic 2-contact metric structure*. M with such a structure is called a *homothetic 2-contact Riemannian manifold*. Especially, if $c_1=c_2=1$ in (2.13), the structure is called a *2-contact metric structure* and M with such a structure is a *2-contact Riemannian manifold*.

If a 2-contact metric structure (resp., homothetic 2-contact metric structure) satisfies

$$L_{\xi_a}\langle\ ,\ \rangle=0 \qquad (a=1,2)$$
(2.14) and
$$\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]+d\eta_1(X,Y)\xi_1+d\eta_2(X,Y)\xi_2=0,$$

then the structure is said to be a *second Sasakian structure* (resp., *homothetic second Sasakian structure*) and M is called a *second Sasakian manifold* (resp., *homothetic second Sasakian manifold*).

The following identities hold on a homothetic second Sasakian manifold:

(2.15) $$\nabla_X\xi_a=c_a\phi X \qquad (a=1,2).$$

(2.16) $$(\nabla_X\phi)Y=c_1\{\eta_1(Y)(X-\eta_2(X)\xi_2)-(\langle X,\ Y\rangle-\eta_2(X)\eta_2(Y))\xi_1\}$$
$$+c_2\{\eta_2(Y)(X-\eta_1(X)\xi_1)-(\langle X,\ Y\rangle-\eta_1(X)\eta_1(Y))\xi_2\}.$$

An almost 2-contact metric structure on M is a homothetic second Sasakian structure if and only if both (2.15) and (2.16) hold.

(2) Submanifolds.

Let $\tilde{M}$ be an m-dimensional Riemannian manifold and M an n-dimensional submanifold in

$\tilde{M}$. Let $\langle\ ,\ \rangle$ be the Riemannian metric on $\tilde{M}$ as well as the induced metric on M. We denote by $\tilde{\nabla}$ the Riemannian connection on $\tilde{M}$ and by ∇ the Riemannian connection on M determined by the induced metric on M. Then the Gauss formula is given by

(2.17) $\quad \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$

for any tangent vector fields X and Y on M, where σ is the second fundamental form on M in $\tilde{M}$.

The Weingarten formula is given by

(2.18) $\quad \tilde{\nabla}_X \alpha = -A_\alpha X + \nabla^\perp_X \alpha$

for any tangent vector field X and normal vector field α on M, where A_α is the second fundamental tensor at α and $\nabla^\perp$ is the normal connection induced in the normal bundle $T^\perp M$. For any tangent vectors X, Y and normal vector α of M, we have

(2.19) $\quad \langle A_\alpha X, Y\rangle = \langle \sigma(X, Y), \alpha\rangle,$

A normal vector field $\mu := \frac{1}{n}$ trace σ is called a *mean curvature vector* of M in $\tilde{M}$.

Let $\tilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold with structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \langle\ ,\ \rangle)$and M an $(n+1)$-dimensional submanifold in $\tilde{M}$. If the structure vector field $\tilde{\xi}$ is tangent to M and the second fundamental form σ of M in $\tilde{M}$ is of form

(2.20) $\quad \sigma(X, Y) = \frac{n+1}{n}(\langle X, Y\rangle - \tilde{\eta}(X)\tilde{\eta}(Y))\mu + \tilde{\eta}(X)\sigma(Y, \tilde{\xi}) + \tilde{\eta}(Y)\sigma(X, \tilde{\xi})$

for any tangent vector fields X and Y on M, M is said to be *totally contact umbilical*. When σ is of form

(2.21) $\quad \sigma(X, Y) = \tilde{\eta}(X)\sigma(Y, \tilde{\xi}) + \tilde{\eta}(Y)\alpha(X, \tilde{\xi})$

for any tangent vector fields X and Y on M, then M is said to be *totally contact geodesic*.

(3) Contact CR-submanifolds.

Let $\tilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold with structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \langle\ ,\ \rangle)$ and M an $(n+1)$-dimensional submanifold in $\tilde{M}$. We assume that the vector field $\tilde{\xi}$ is tangent to M. For simplicity, put $\xi := \tilde{\xi}$ on M and $\eta(X) := \tilde{\eta}(X)$ for any tangent vector X of M, and with $\langle\xi\rangle$ the 1-dimensional distribution spanned by ξ on M; then M is called a *contact CR-submanifold* in $\tilde{M}$ if there exist two differentiable distributions D and $D^\perp$ on M satisfying

(2.22) $\quad TM = D \oplus D^\perp \oplus \langle\xi\rangle$

where D, $D^\perp$ and $\langle\xi\rangle$ are mutually orthogonal,

(2.23) $\quad \tilde{\phi}(D_x) = D_x \qquad$ for each $x \in M$

and

(2.24) $\quad \tilde{\phi}(D^\perp_x) \subset T^\perp_x M \qquad$ for each $x \in M$.

Then dim D is even, and the normal vector space $T^\perp_x M$ is split as $T^\perp_x M = \tilde{\phi}(D^\perp_x) \oplus N_x$ for any point $x \in M$, where N_x is the orthogonal complement of $\tilde{\phi}(D^\perp_x)$ in $T^\perp_x M$.

For the totally contact umbilical contact CR-submanifold M in $\tilde{M}$, we see the following equations:

$$\tilde{\nabla}_X Y = \nabla_X Y + k\langle X, Y\rangle \mu,$$

(2.25) $\quad \tilde{\nabla}_X \alpha = -k\langle \alpha, \mu\rangle X + \nabla^\perp_X \alpha,$

$$\langle \tilde{R}(X, Y)Z, \alpha\rangle = \langle Y, Z\rangle\langle \nabla^\perp_X \mu, \alpha\rangle - \langle X, Z\rangle\langle \nabla^\perp_Y \mu, \alpha\rangle$$

for any $X, Y, Z \in D \oplus D^{\perp}$ and normal vector field α on M, where $k=\frac{n+1}{n}$;

(2.26) $\sigma(X,\xi)=0$ and $\nabla_X \xi=\tilde{\phi}X$ for any $X \in D$;

(2.27) $\sigma(V,\xi)=\tilde{\phi}V$ for any $V \in D^{\perp}$.

§ 3. Totally contact umbilical contact CR-submanifolds.

LEMMA 1. *Let M be a totally contact umbilical contact CR-submanifold in a Sasakian manifold $\tilde{M}$. If M is not anti-invariant,*
(i.e., $D \neq \{0\}$), then it follows that

(3.1) $\mu \in \tilde{\phi}(D^{\perp})$

and

(3.2) $\dim D^{\perp}=1$ *or* *M is totally contact geodesic.*

PROOF. For any non-zero $X \in D$, we have

$$\begin{aligned} 0=(\tilde{\nabla}_X \tilde{\phi})\mu &= \tilde{\nabla}_X \tilde{\phi}\mu - \tilde{\phi}\tilde{\nabla}_X \mu \\ &= \tilde{\nabla}_X \tilde{\phi}\mu_1 + \tilde{\nabla}_X \tilde{\phi}\mu_2 + k \| \mu \|^2 \tilde{\phi}X - \tilde{\phi}\nabla^{\perp}_X \mu \\ &= \tilde{\nabla}_X \tilde{\phi}\mu_1 - k\langle \tilde{\phi}\mu_2, \mu\rangle X + \nabla^{\perp}_X \phi\mu_2 + k \| \mu \|^2 \tilde{\phi}X - \phi\nabla^{\perp}_X \mu, \end{aligned}$$

where μ_1 is $\tilde{\phi}(D^{\perp})$-component of μ, $\mu_2 := \mu - \mu_1$ and $k := \frac{n+1}{n}$, from which we get

$$\begin{aligned} 0 &= \langle \tilde{\nabla}_X \tilde{\phi}\mu_1, \tilde{\phi}X\rangle + k \| \mu \|^2 \| X \|^2 - \langle \tilde{\phi}\nabla^{\perp}_X \mu, \tilde{\phi}X\rangle \\ &= X\langle \tilde{\phi}\mu_1, \tilde{\phi}X\rangle - \langle \tilde{\phi}\mu_1, \tilde{\nabla}_X \tilde{\phi}X\rangle + k \| \mu \|^2 \| X \|^2 \\ &= X(\langle \mu_1, X\rangle - \tilde{\eta}(\mu_1)\tilde{\eta}(X)) - \langle \tilde{\phi}\mu_1, \tilde{\nabla}_X \tilde{\phi}X\rangle + k \| \mu \|^2 \| X \|^2 \\ &= -\langle \tilde{\phi}\mu_1, (\tilde{\nabla}_X \tilde{\phi})X + \tilde{\phi}\tilde{\nabla}_X X\rangle + k \| \mu \|^2 \| X \|^2 \\ &= -\langle \tilde{\phi}\mu_1, \tilde{\eta}(X)X - \| X \|^2 \xi + \tilde{\phi}\nabla_X X + k \| X \|^2 \tilde{\phi}\mu\rangle + k \| \mu \|^2 \| X \|^2 \\ &= k(\| \mu \|^2 - \| \mu_1 \|^2) \| X \|^2. \end{aligned}$$

This proves (3.1).

Next, for any $V, W \in D^{\perp}$, we have

$$\begin{aligned} \eta(W)V - \langle V, W\rangle \xi &= (\tilde{\nabla}_V \tilde{\phi})W \\ &= \tilde{\nabla}_V \tilde{\phi}W - \tilde{\phi}\tilde{\nabla}_V W \\ &= k\langle \tilde{\phi}\mu, W\rangle V + \nabla^{\perp}_V \tilde{\phi}W - \tilde{\phi}\nabla_V W - k\langle V, W\rangle \tilde{\phi}\mu. \end{aligned}$$

Taking the inner product with V, we get

(3.3) $\langle \tilde{\phi}\mu, W\rangle \| V \|^2 = \langle V, W\rangle \langle \tilde{\phi}\mu, V\rangle$.

Interchanging the role of V and W in the above equation, we have

(3.4) $\langle \tilde{\phi}\mu, V\rangle \| W \|^2 = \langle V, W\rangle \langle \tilde{\phi}\mu, W\rangle$

From (3.3) and (3.4), we obtain

(3.5) $\langle V, \tilde{\phi}\mu\rangle \{ \| W \|^2 \| V \|^2 - \langle V, W\rangle^2 \} = 0$

If $\dim D^{\perp} \geq 2$, $\langle V, \tilde{\phi}\mu\rangle = 0$ for any $V \in D^{\perp}$, from which we have $\mu = 0$ identically. Q.E.D.

LEMMA 2. *Let M be a totally contact umbilical contact CR-submanifold in a Sasakian manifold $\tilde{M}$. If M is not anti-invariant and* $\dim M \geq 6$, *then it follows that*

$$\nabla^{\perp}_X \mu = 0$$

for any vector field X on M such that $X \perp \xi$.

PROOF. Assume that $\mu\neq 0$. By Lemma 1, we have $\mu\in\tilde{\phi}(D^{\perp})$and dim $D^{\perp}=1$. For any $X\in D$, there exists a $Y\in D(Y\neq 0)$ such that $\langle X,Y\rangle=\langle X,\phi Y\rangle=0$, since dim $D\geq 4$. From (2.9) and (2.25), we have

$$\| Y \|^2\langle\nabla^{\perp}_X\mu,\alpha\rangle=\langle\tilde{R}(X,Y)Y,\alpha\rangle=\langle\tilde{R}(\tilde{\phi}X,\tilde{\phi}Y)Y,\alpha\rangle=0$$

for any normal vector field α on M, from which

(3.6) $\quad\nabla^{\perp}_X\mu=0\qquad$ for any $X\in D$.

Since $\tilde{\phi}\alpha\in N$ for any $\alpha\in N$, we have

$$\| X \|^2\langle\nabla^{\perp}_V\mu,\alpha\rangle=\langle\tilde{R}(V,X)X,\alpha\rangle=\langle\tilde{R}(V,X)\tilde{\phi}X,\tilde{\phi}\alpha\rangle=0$$

for any $V\in D^{\perp}$ and non-zero $X\in D$, from which

(3.7) $\quad\nabla^{\perp}_V\mu\in\tilde{\phi}(D^{\perp})$

for any $V\in D^{\perp}$.

From (2.9) and (2.25), we have

$$\langle\tilde{R}(V,X)X,V\rangle=\| V \|^2\| X \|^2$$

for any $X\in D$ and $V\in D^{\perp}$, from which we obtain

$$\begin{aligned}&\| X \|^2\langle\nabla^{\perp}_V\mu,\tilde{\phi}V\rangle\\&=\langle\tilde{R}(V,X)\tilde{\phi}X,V\rangle\\&=0.\end{aligned}$$

Therefore $\nabla^{\perp}_V\mu\in N$ for any $V\in D^{\perp}$. Hence it follows that $\nabla^{\perp}_X\mu=0$ for any tangent vector field X on M such that $X\perp\xi$. Q.E.D.

LEMMA 3. *Let M be a totally contact umbilical contact CR-submanifold in a Sasakian manifold $\tilde{M}$. If M is not anti-invariant, then we have*

(3.8) $\quad\nabla^{\perp}_{\xi}\mu\in\tilde{\phi}(D^{\perp})$.

PROOF. Assume that $\mu\neq 0$. Since $\tilde{\nabla}_{\xi}\tilde{\phi}\mu=\tilde{\phi}\tilde{\nabla}_{\xi}\mu$ and $\sigma(V_0,\xi)=\mu$, we have

$$\nabla_{\xi}V_0+\mu=\tilde{\phi}A_{\mu}\xi-\tilde{\phi}\nabla^{\perp}_{\xi}\mu,$$

where $V_0\in D^{\perp}$ such that $\tilde{\phi}V_0=\mu$.

Taking the inner product with any $\alpha\in N$, we get

$$\begin{aligned}0&=\langle\tilde{\phi}A_{\mu}\xi,\alpha\rangle-\langle\tilde{\phi}\nabla^{\perp}_{\xi}\mu,\alpha\rangle\\&=-\langle\tilde{\phi}\nabla^{\perp}_{\xi}\mu,\alpha\rangle\\&=\langle\nabla^{\perp}_{\xi}\mu,\tilde{\phi}\alpha\rangle.\end{aligned}$$

Since $\tilde{\phi}(N)=N$, we obtain $\nabla^{\perp}_{\xi}\mu\in\phi(D^{\perp})$. Q.E.D.

LEMMA 4. *Let M be a totally contact umbilical contact CR-submanifold in a Sasakian manifold $\tilde{M}$. If M is not anti-invariant and* dim $M\geq 6$, *then it follows that*

(3.9) $\quad\nabla_W V_0=0$

for any $W\in D^{\perp}$, where $V_0\in D^{\perp}$ such that $\tilde{\phi}V_0=\mu$.

PROOF. For any $W\in D^{\perp}$, by Lemma 2,

$$\begin{aligned}-A_{\mu}W&=-A_{\mu}W+\nabla^{\perp}_W\mu\\&=\tilde{\nabla}_W(\tilde{\phi}V_0)\\&=-\langle W,V_0\rangle\xi+\tilde{\phi}\nabla_W V_0+k\langle V_0,W\rangle\tilde{\phi}\mu,\end{aligned}$$

where $k=\dfrac{n+1}{n}$.

Since $A_{\mu}W=k\| \mu \|^2W+\langle\tilde{\phi}W,\mu\rangle\xi=k\| \mu \|^2W+\langle W,V_0\rangle\xi$, *we have*

$$k\| \mu \|^2W+\tilde{\phi}\nabla_W V_0+k\langle W,V_0\rangle\tilde{\phi}\mu=0,$$

from which we obtain $\tilde{\phi}\nabla_W V_0 \in D^\perp$ by virtue of Lemma 1.
Thus we get $\nabla_W V_0 = 0$. Q.E.D.

THEOREM 1. *Let M be a totally contact umbilical contact CR-submanifold in a Sasakian manifold $\tilde{M}$. If M is not anti-invariant and* dim $M \geq 6$, *then μ is parallel.*

PROOF. It is enough to assume that M is not totally contact geodesic. Since M is not totally contact geodesic, by Lemma 1, we have dim $D^\perp = 1$ *and* $\mu \in \tilde{\phi}(D^\perp)$. By (2.9) and the Codazzi equation, we have

$$(\bar{\nabla}_\xi \sigma)(W, V_0) = (\bar{\nabla}_W \sigma)(\xi, V_0),$$

for any $W \in D^\perp$, where $(\bar{\nabla}_X \sigma)(Y, Z) = \nabla^\perp_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - (Y, \nabla_X Z)$ and $\tilde{\phi} V_0 = \mu$. Since $\nabla_W V_0 = 0$ by Lemma 4 and $\nabla^\perp_W \mu = 0$ by Lemma 2,

$$(\bar{\nabla}_W \sigma)(\xi, V_0) = \nabla^\perp_W \sigma(\xi, V_0) = \nabla^\perp_W \mu = 0,$$

from which we get

$$\begin{aligned} & k \langle W, V_0 \rangle \nabla^\perp_\xi \mu \\ &= (\bar{\nabla}_\xi \sigma)(W, V_0) \\ &= (\bar{\nabla}_W \sigma)(\xi, V_0) \\ &= 0. \end{aligned}$$

Putting $W = V_0$, we obtain

$$\| V_0 \|^2 \nabla^\perp_\xi \mu = 0.$$

Therefore $\| \mu \|^2 \| \nabla^\perp_\xi \mu \|^2 = 0$.
Since $\mu \in \tilde{\phi}(D^\perp)$, $\nabla^\perp_\xi \mu \in \tilde{\phi}(D^\perp)$ and dim $D^\perp = 1$, we have

$$\frac{1}{2} \xi(\| \mu \|^2) = \langle \mu, \nabla^\perp_\xi \mu \rangle = \| \mu \| \| \nabla^\perp_\xi \mu \| = 0.$$

From this and Lemma 2, μ is (non-zero) constant. Therefore μ is parallel. Q.E.D.

§ 4. Totally contact umbilical submanifolds with non-zero parallel mean curvature vector.

THEOREM 2 (cf., [3]). *Let M be a totally contact umbilical submanifold with non-zero parallel mean curvature vector μ in a Sasakian manifold $\tilde{M}$. If $\tilde{\phi}\mu$ is tangent to M, then M is a contact CR-submanifold with $D^\perp = \langle \tilde{\phi}\mu \rangle$. Furthermore M admits a homothetic second Sasakian structure.*

PROOF. Since the mean curvature vector μ is non-zero parallel, the mean curvature $c := \sqrt{\langle \mu, \mu \rangle}$ is a non-zero constant. For simplicity, we put $\xi := \tilde{\xi}$ on M and $\eta(X) := \tilde{\eta}(X)$ for any tangent vector X of M.

We put $\hat{\xi} := -\frac{1}{c} \tilde{\phi}\mu$. Then $\hat{\xi}$ is a unit tangent vector field on M and $\eta(\hat{\xi}) = 0$.

For any tangent vector field X on M, we put

$$\tilde{\phi}X = \phi X + \phi' X + \frac{1}{c} \langle X, \hat{\xi} \rangle \mu \tag{4.1}$$

where ϕX is the tangential part of $\tilde{\phi}X$ and $\phi' X$ is the normal part of $\tilde{\phi}X$ orthogonal to μ.

We obtain the following equations:

$$\begin{aligned} & \nabla_X \xi = \phi X, \\ & \nabla_X \hat{\xi} = \frac{(n+1)c}{n} \phi X, \end{aligned} \tag{4.2}$$

$$(\nabla_X\phi)Y=\frac{(n+1)c}{n}\{\langle Y,\hat{\xi}\rangle(X-\eta(X)\xi)-(\langle X,Y\rangle-\eta(X)\eta(Y))\hat{\xi}\}$$
$$+\langle\phi^2X,Y\rangle\xi-\eta(Y)\phi^2X.$$

Using these equations, we get

$$\begin{aligned}&\langle\phi^2X,Y\rangle+\langle X,Y\rangle-\eta(X)\eta(Y)-\langle X,\hat{\xi}\rangle\langle Y,\hat{\xi}\rangle\\&=-\frac{n}{(n+1)c}\{\langle(\nabla_X\phi)Y,\hat{\xi}\rangle+\langle\phi Y,\nabla_X\hat{\xi}\rangle\}\\&=-\frac{n}{(n+1)c}\{X\langle\phi Y,\hat{\xi}\rangle-\langle\phi(\nabla_X Y),\hat{\xi}\rangle\}\\&=-\frac{n}{(n+1)c^2}\{X\langle Y,\mu\rangle-\langle\nabla_X Y,\mu\rangle\}\\&=0,\end{aligned}$$

from which

(4.3) $\phi^2X=-X+\eta(X)\xi+\langle X,\hat{\xi}\rangle\hat{\xi}$ for any $X\in TM$.

Then we obtain

(4.4) $\tilde{\phi}\hat{\xi}=\frac{1}{c}\mu$

and

(4.5) $\tilde{\phi}X=\phi X$

for any tangent vector X of M orthogonal to $\hat{\xi}$. This shows that M is a contact CR-submanifold with $D^{\perp}=\langle\tilde{\phi}\mu\rangle$.

Next, we put $\hat{\eta}(X):=\langle X,\hat{\xi}\rangle$. Then we have

$$\nabla_X\xi=\phi X\quad,\quad\nabla_X\hat{\xi}=\frac{(n+1)c}{n}\phi X,$$

(4.6) $$(\nabla_X\phi)Y=\frac{(n+1)c}{n}\{\hat{\eta}(Y)(X-\eta(X)\xi)-(\langle X,Y\rangle-\eta(X)\eta(Y))\hat{\xi}\}$$
$$+\eta(Y)(X-\hat{\eta}(X)\hat{\xi})-(\langle X,Y\rangle-\hat{\eta}(X)\hat{\eta}(Y))\xi.$$

Therefore $(\phi,\xi,\hat{\xi},\eta,\hat{\eta},\langle\ ,\ \rangle)$ is a homothetic second Sasakian structure. Q.E.D.

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