Cyclic Parallel Hypersurfaces in a Sasakian Space Form

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佐々木空間形の巡回平行超曲面

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Abstract

We investigate cyclic parallel hypersurfaces in a Sasakian space form and prove some theorems.

§1. Introduction.

Let M be a submanifold in a Riemannian manifold $\tilde{M}$. If the second fundamental form σ of M in $\tilde{M}$ is cyclic parallel, that is,

$$(\bar{\nabla}_X \sigma)(Y, Z) + (\bar{\nabla}_Y \sigma)(Z, X) + (\bar{\nabla}_Z \sigma)(X, Y) = 0$$

for arbitrary vectors X, Y, Z tangent to M, then M is said to be cyclic parallel.

Recently, U-H. Ki [5] has proved that a real hypersurface M in a real $2m(\geq 4)$-dimensional complex space form $\tilde{M}(c)$ with nonzero constant holomorphic sectional curvature c is cyclic parallel if and only if $\varphi A = A\varphi$, where φ denotes the structure tensor induced on M by almost complex structure of $\tilde{M}(c)$ and A the second fundamental tensor derived from the unit normal.

In this paper, we investigate cyclic parallel hypersurfaces in a Sasakian space form and prove the following theorems :

THEOREM 1. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c \neq 1$, of dimension $2m+1(\geq 5)$. Then the structure vector field ξ of $\tilde{M}(c)$ is tangent to M.*

THEOREM 2. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$, of dimension $2m+1(\geq 5)$. Then the structure tensor φ induced on M and the second fundamental tensor A derived from the unit normal commute into each other, that is, $\varphi A=A\varphi$.*

THEOREM 9. *Let M be a hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq -3$, of dimension $2m+1(\geq 5)$. If the structure tensor φ induced on M and the second fundamental tensor A derived from the unit normal commute into each other, then M is cyclic parallel.*

THEOREM 10. *Let M be a complete hypersurface with $\varphi A=A\varphi$ in a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$, where φ is the structure tensor induced on M, and A the second fundamental tensor derived from the unit normal. If the structure vector field $\tilde{\xi}$ of $\tilde{M}(c)$ is not tangent to M evrywhere on M, then M is a totally umbilical hypersurface with constant mean curvature, isometric to an ordinary sphere, and $c=1$.*

Throughout this paper, we assume that all objects under consideration are differentiable of class C^{∞} and that all manifolds are connected unless otherwise stated.

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§2. Preliminaries.

This section introduces some definitions and the fundamental properties used throughout the paper.

(1) Let M be a $(2m+1)$-dimensional Sasakian manifold. We denote by $(\varphi, \xi, \eta, \langle\ ,\ \rangle)$ The Sasakian structure of M, where φ is a tensor field of type (1, 1), ξ a vector field, η a 1-form and $\langle\ ,\ \rangle$ a Riemannian metric. The structure tensors satisfy the following equations :

$$\begin{aligned}&\varphi^2X=-X+\eta(X)\xi,\quad \varphi\xi=0,\quad \eta(\varphi X)=0,\quad \eta(\xi)=1,\\ &\langle\varphi X, Y\rangle+\langle X, \varphi Y\rangle=0,\quad \eta(X)=\langle\xi, X\rangle,\\ &\nabla_X\xi=\varphi X,\quad (\nabla_X\varphi)Y=\eta(Y)X-\langle X, Y\rangle\xi\end{aligned}\tag{2.1}$$

for any vector fields X, Y tangent to M, where ∇ denotes the Riemannian connection of M.

The Riemannian curvature tensor R of the Sasakian manifold M, defined by $R(X, Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$, satisfies

$$\begin{aligned}&R(\varphi X, \varphi Y)Z=R(X, Y)Z-\langle Y, Z\rangle X+\langle X, Z\rangle Y-\langle Y, \varphi Z\rangle\varphi X+\langle X, \varphi Z\rangle\varphi Y,\\ &R(X, \varphi Y)Z=-R(\varphi X, Y)Z-\langle Y, \varphi Z\rangle X+\langle X, \varphi Z\rangle Y+\langle Y, Z\rangle\varphi X-\langle X, Z\rangle\varphi Y\\ &\text{and}\\ &R(X, Y)\xi=\eta(Y)X-\eta(X)Y.\end{aligned}\tag{2.2}$$

A Sasakian manifold M is called a Sasakian space form if M is of constant φ-holomorphic sectional curvature. The Riemannian curvature tensor R of the Sasakian space form $M(c)$ of constant φ-holomorphic sectional curvature c takes the following form :

$$\begin{aligned} R(X, Y)Z=&\frac{c+3}{4}\{\langle Y, Z\rangle X-\langle X, Z\rangle Y\}+\frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X \\ &+\langle X, Z\rangle\eta(Y)\xi-\langle Y, Z\rangle\eta(X)\xi+\langle X, \varphi Z\rangle\varphi Y \\ &-\langle Y, \varphi Z\rangle\varphi X+2\langle X, \varphi Y\rangle\varphi Z\}. \end{aligned} \tag{2.3}$$

(2) Let $\tilde{M}$ be a Riemannian manifold and M a Riemannian manifold isometrically immersed in $\tilde{M}$. Then M is called a submanifold in $\tilde{M}$. Particularly, M is called a hypersurface in $\tilde{M}$ if codim $M=1$. The Riemannian meteic on $\tilde{M}$ as well as the induced metric on M is denoted by $\langle\ ,\ \rangle$. Let ∇ and $\tilde{\nabla}$ be the Riemannian connections on M and $\tilde{M}$, respectively. Then the Gauss and Weingarten formulas are given by

(2.4) $\tilde{\nabla}_X Y=\nabla_X Y+\sigma(X, Y)$

and

(2.5) $\tilde{\nabla}_X N=-A_N X+\nabla_X^\perp N$

for any vector fields X, Y tangent to M and N normal to M, where σ denotes the second fundamental form, A_N the second fundamental tensor at N and $\nabla^\perp$ the linear connection induced in the normal bundle $T^\perp M$, called the normal connection. The second fundamental tensor A_N is related to the second fundamental form σ by

(2.6) $\langle A_N X, Y\rangle=\langle\sigma(X, Y), N\rangle$.

Denoting the Riemannian curvature tensors of M and $\tilde{M}$ by R and $\tilde{R}$ respectively, the equations of Gauss and Codazzi are given by

(2.7) $\langle W, R(X, Y)Z\rangle=\langle W, \tilde{R}(X, Y)Z\rangle+\langle\sigma(W, X), \sigma(Y, Z)\rangle-\langle\sigma(W, Y), \sigma(X, Z)\rangle$

and

(2.8) $\langle\tilde{R}(X, Y)Z, N\rangle=\langle(\bar{\nabla}_X\sigma)(Y, Z), N\rangle-(\bar{\nabla}_Y\sigma)(X, Z), N\rangle$

for any vectors W, X, Y, Z tangent to M and N normal to M, where the first covariant differentiation $\bar{\nabla}\sigma$ of σ is defined by

(2.9) $(\bar{\nabla}_X\sigma)(Y, Z)=\nabla_X^\perp\sigma(Y, Z)-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z)$.

A submanifold M is called a parallel submanifold if σ is parallel, i. e., $\bar{\nabla}\sigma=0$ identically. A submanifold M is called a cyclic parallel submanifold if the cyclic sum of $(\bar{\nabla}_X\sigma)(Y, Z)$ vanishes identically, i. e.,

(2.10) $(\bar{\nabla}_X\sigma)(Y, Z)+(\bar{\nabla}_Y\sigma)(Z, X)+(\bar{\nabla}_Z\sigma)(X, Y)=0$

for any vectors X, Y, Z tangent to M. It is easily seen that condition (2.10) is equivalent to

(2.11) $(\bar{\nabla}_X\sigma)(X, X)=0$ for all $X\in TM$.

It was proved in [2] that any geodesic hypersphere in a complex space form with non-zero constant holomorphic sectional curvature is cyclic parallel and not parallel.

Let M be submanifold in a Sasakian manifold $\tilde{M}$ with structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \langle\ ,\ \rangle)$. M is said to be anti-invariant if

(2.12) $\tilde{\varphi}(T_x M)\subset T_x M$ for each $x\in M$.

We have the following well-known lemma :

LEMMA 1 (e. g., see [9]). *Let M be an n-dimensional submanifold in a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$. If the structure vector field $\tilde{\xi}$ is normal to M, then M is anti*

-invariant, and $m \geq n$.

(3) Let M be a hypersurface in a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$ with Sasakian structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}\ \langle\ ,\ \rangle)$. A unit normal ε to M may then be chosen. For this unit normal ε, we put

$$
\begin{aligned}
&f := \tilde{\eta}(\varepsilon),\ \zeta := -\varphi\varepsilon,\ \xi := \tilde{\xi} - f\varepsilon,\ \varphi X := \tilde{\varphi}X - \langle \zeta, X\rangle \varepsilon \\
&AX := A_\varepsilon X \text{ and } h\langle X, Y\rangle := \langle AX, Y\rangle = \langle \sigma(X, Y), \varepsilon\rangle
\end{aligned}
\tag{2. 13}
$$

for any vectors X, Y tangent to M. By the properties of the Sasakian structure, the following relations are given :

$$
\begin{aligned}
&\langle \zeta, \xi\rangle = 0,\quad \|\zeta\|^2 = \|\xi\|^2 = 1 - f^2,\quad \varphi\zeta = -f\xi,\quad \varphi\xi = f\zeta, \\
&\varphi^2 X = -X + \langle \zeta, X\rangle \zeta + \eta(X)\xi,\quad \langle \varphi X, Y\rangle = -\langle X, \varphi Y\rangle \\
&\nabla_X \zeta = \varphi AX - f X,\quad \nabla_X \xi = \varphi X + fAX,\quad Xf = \langle \zeta - A\xi, X\rangle, \\
&(\nabla_X \varphi) Y = \langle \zeta, Y\rangle AX - \langle AX, Y\rangle \zeta + \eta(Y) X - \langle X, Y\rangle \xi
\end{aligned}
\tag{2.14}
$$

for any vectors X, Y tangent to M, where $\eta(X) := \langle \xi, X\rangle$.

A scalar function $\rho := \frac{1}{2m}$trace A is called a mean curvature of M in $\tilde{M}$. M is said to be totally umbilical if $AX = \rho X$ for any vector X tangent to M. Particularly, M is said to be totally geodesic, if $AX = 0$ for any vector X tangent to M. If the structure vector field $\tilde{\xi}$ is tangent to M and

$$
AX = \frac{2m}{2m-1}\rho(X - \eta(X)\xi) + \eta(X)A\xi + \langle A\xi, X\rangle \xi
\tag{2.15}
$$

for any vector X tangent to M, then M is said to be totally contact umbilical. When a totally contact umbilical hypersurface M has vanishing mean curvature, then M is said to be totally contact geodesic.

In the following, the ambient Sasakian manifold is assumed to be a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1$. Then the equations of Gauss and Codazzi for M in $\tilde{M}(c)$ are respectively rewritten as :

$$
\begin{aligned}
R(X, Y)Z = &\frac{c+3}{4}(\langle Y, Z\rangle X - \langle X, Z\rangle Y) + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
&+ \langle X, Z\rangle \eta(Y)\xi - \langle Y, Z\rangle \eta(X)\xi + \langle X, \varphi Z\rangle \varphi Y - \langle Y, \varphi Z\rangle \varphi X \\
&+ 2\langle X, \varphi Y\rangle \varphi Z\} + \langle AY, Z\rangle AX - \langle AX, Z\rangle AY.
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = &\frac{c-1}{4}\{f(\eta(Y)X - \eta(X)Y) + \langle \zeta, X\rangle \varphi Y - \langle \zeta, Y\rangle \varphi X \\
&+ 2\langle X, \varphi Y\rangle \zeta\},
\end{aligned}
\tag{2.17}
$$

i. e.,

$$
\begin{aligned}
(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = &\frac{c-1}{4}\{f(\eta(Y)\langle X, Z\rangle - \eta(X)\langle Y, Z\rangle) \\
&+ \langle \zeta, Y\rangle \langle X, \varphi Z\rangle - \langle \zeta, X\rangle \langle Y, \varphi Z\rangle + 2\langle \zeta, Z\rangle \langle X, \varphi Y\rangle\}.
\end{aligned}
$$

LEMMA 2. *Let M be a hypersurface in a Sasakian space form $\tilde{M}(c)$. Then M is cyclic parallel if and only if*

(2.18)
$$(\nabla_X h)(Y, Z)=\frac{c-1}{4}(\langle\xi, Y\rangle\langle X, \varphi Z\rangle+\langle\xi, Z\rangle\langle X,\varphi Y\rangle)$$
$$+\frac{c-1}{12} f(\langle X, Y\rangle\eta(Z)+\langle X, Z\rangle\eta(Y)-2\langle Y, Z\rangle\eta(X))$$

for any vectors X, Y, Z tangent to M.

The proof for Lemma 2 is simple and has been omitted.

Lemma 2 simply leads to the following

REMARK. *If there exists a parallel hypersurface in a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$, then $c=1$.*

It is known that a totally umbilical hypersurface M in a Riemannian manifold $\tilde{M}$ is parallel if and only if the mean curvature ρ of M in $\tilde{M}$ is a constant. For Sasakian geometry, we have

LEMMA 3 [4, 8]. *Let M be a totally umbilical hypersurface in a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$. Then $c=1$ and M is parallel.*

§3. Cyclic parallel hypersurface in a Sasakian space form.

Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$. By Lemma 2, we have

(3. 1)
$$(\nabla h)(X, Y, Z):=(\nabla_X h)(Y, Z)$$
$$=\frac{c-1}{4}(\langle\xi, Y\rangle\langle X, \varphi Z\rangle+\langle\xi, Z\rangle\langle X, \varphi Y\rangle)$$
$$+\frac{c-1}{12}f(\langle X, Y\rangle\eta(Z)+\langle X, Z\rangle\eta(Y)-2\langle Y, Z\rangle\eta(X))$$

for any vectors X, Y, Z tangent to M. By differentiating this covariantly along M and making use of (2. 14), we find

(3. 2)
$$(\nabla\nabla h)(W, X, Y, Z):=(\nabla_W(\nabla h))(X, Y, Z)$$
$$=\frac{c-1}{4}\{(\langle AW, X\rangle\langle\xi, Y\rangle-\langle AW, Y\rangle\langle\xi, X\rangle+\langle W, X\rangle\eta(Y)-\langle W, Y\rangle\eta(X))\langle\xi, Z\rangle$$
$$+(\langle AW, X\rangle\langle\xi, Z\rangle-\langle AW, Z\rangle\langle\xi, X\rangle+\langle W, X\rangle\eta(Z)-\langle W, Z\rangle\eta(X))\langle\xi, Y\rangle$$
$$+\langle X, \varphi Y\rangle(\langle\varphi AW, Z\rangle-f\langle W, Z\rangle)+\langle X,\varphi Z\rangle(\langle\varphi AW, Y\rangle-f\langle W, Y\rangle)\}$$
$$+\frac{c-1}{12}\langle\xi-A\xi, W\rangle(\langle X, Y\rangle\eta(Z)+\langle X, Z\rangle\eta(Y)-2\langle Y, Z\rangle\eta(X))$$
$$+\frac{c-1}{12}f\{\langle X, Y\rangle(\langle\varphi W, Z\rangle+f\langle AW, Z\rangle)+\langle X, Z\rangle(\langle\varphi W, Y\rangle+f\langle AW, Y\rangle)$$
$$-2\langle Y, Z\rangle(\langle\varphi W, X\rangle+f\langle AW, X\rangle)\}$$

for any vectors W, X, Y, Z tangent to M.

Substituting this and (2. 16) into the Ricci formula given by

$$(\nabla\nabla h)(W,X,Y,Z)-(\nabla\nabla h)(X,W,Y,Z)=-\langle R(W,X)Y,AZ\rangle-\langle R(W,X)Z,AY\rangle,$$

it follows that

$$\begin{aligned}
&\langle AW,Y\rangle\langle A^2X,Z\rangle+\langle AW,Z\rangle\langle A^2X,Y\rangle-\langle AX,Y\rangle\langle A^2W,Z\rangle-\langle AX,Z\rangle\langle A^2W,Y\rangle\\
&=(\frac{c+3}{4}+\frac{c-1}{12}f^2)(\langle AW,Y\rangle\langle X,Z\rangle+\langle AW,Z\rangle\langle X,Y\rangle-\langle AX,Y\rangle\langle W,Z\rangle\\
&\qquad-\langle AX,Z\rangle\langle W,Y\rangle)\\
&\quad-\frac{c-1}{4}\{(\langle AW,Y\rangle\langle\xi,X\rangle-\langle AX,Y\rangle\langle\xi,W\rangle+\langle W,Y\rangle\eta(X)-\langle X,Y\rangle\eta(W))\langle\xi,Z\rangle\\
&\qquad+\langle AW,Z\rangle\langle\xi,X\rangle-\langle AX,Z\rangle\langle\xi,W\rangle+\langle W,Z\rangle\eta(X)-\langle X,Z\rangle\eta(W))\langle\xi,Y\rangle\\
&\qquad-\langle(\varphi A-A\varphi)W,Y\rangle\langle X,\varphi Z\rangle+\langle(\varphi A-A\varphi)X,Y\rangle\langle W,\varphi Z\rangle\\
&\qquad-\langle(\varphi A-A\varphi)W,Z\rangle\langle X,\varphi Y\rangle+\langle(\varphi A-A\varphi)X,Z\rangle\langle W,\varphi Y\rangle\\
&\qquad+2\langle(\varphi A-A\varphi)Y,Z\rangle\langle W,\varphi X\rangle\\
&\qquad+\langle AW,Y\rangle\eta(X)\eta(Z)-\langle AX,Y\rangle\eta(W)\eta(Z)+\langle AW,Z\rangle\eta(X)\eta(Y)\\
&\qquad-\langle AX,Z\rangle\eta(W)\eta(Y)+\eta(W)\langle X,Y\rangle\langle A\xi,Z\rangle-\eta(X)\langle W,Y\rangle\langle A\xi,Z\rangle\\
&\qquad+\eta(W)\langle X,Z\rangle\langle A\xi,Y\rangle-\eta(X)\langle W,Z\rangle\langle A\xi,Y\rangle\}\\
&\quad+\frac{c-1}{6}f(\langle W,\varphi Y\rangle\langle X,Z\rangle-\langle X,\varphi Y\rangle\langle W,Z\rangle+\langle W,\varphi Z\rangle\langle X,Y\rangle\\
&\qquad-\langle X,\varphi Z\rangle\langle W,Y\rangle+2\langle W,\varphi X\rangle\langle Y,Z\rangle)\\
&\quad+\frac{c-1}{12}\langle\xi-A\xi,W\rangle\ (\eta(Y)\langle X,Z\rangle+\eta(Z)\langle X,Y\rangle-2\eta(X)\langle Y,Z\rangle)\\
&\quad-\frac{c-1}{12}\langle\xi-A\xi,X\rangle\ (\eta(Y)\langle W,Z\rangle+\eta(Z)\langle W,Y\rangle-2\eta(W)\langle Y,Z\rangle)
\end{aligned}\tag{3.3}$$

for any vectors W, X, Y, Z tangent to M.

THEOREM 1. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$, of dimension $2m+1$ (≥ 5). Then the structure vector field $\tilde{\xi}$ of $\tilde{M}(c)$ is tangent to M.*

PROOF. Let $\{E_1,\cdots\cdots, E_{2m}\}$ be an orthonormal basis of T_xM for any point $x\in M$. Substituting $Y=Z=E_i$ into (3.3) and summing up i from 1 to $2m$, we have

$$(c-1)\{2f\varphi X-\eta(X)(\xi-A\xi)+\langle\xi-A\xi,X\rangle\xi\}=0$$

from which

$$2fX=3f(\langle\xi,X\rangle\xi+\eta(X)\xi)-f\langle A\xi,X\rangle\xi+\eta(X)\varphi A\xi \tag{3.4}$$

for any tangent vector X of M, because of $c\neq 1$.

Substituting $W=Z=E_i$ into (3. 3) and summing up i from 1 to $2m$, we obtain

$$\begin{aligned}
&(\mathrm{tr}\ A)\langle A^2X,Y\rangle-(\mathrm{tr}\ A^2)\langle AX,Y\rangle\\
&=(\frac{c+3}{4}+\frac{c-1}{12}f^2)((\mathrm{tr}\ A)\langle X,Y\rangle-2m\langle AX,Y\rangle)-(c-1)\langle\varphi A\varphi X,Y\rangle\\
&\quad-\frac{c-1}{3}\beta\langle X,Y\rangle+\frac{c-1}{2}(\langle A\xi,X\rangle\langle\xi,Y\rangle+\langle A\xi,Y\rangle-(1+f^2)\langle AX,Y\rangle)\\
&\quad+\frac{c-1}{4}(\mathrm{tr}\ A)(\langle\xi,X\rangle\langle\xi,Y\rangle+\eta(X)\eta(Y))+\frac{(c-1)(m+1)}{3}f\langle\varphi X,Y\rangle
\end{aligned}\tag{3.5}$$

$$+\frac{(c-1)(3m+4)}{6}\eta(X)\langle A\xi, Y\rangle+\frac{(c-1)(m+2)}{6}\eta(Y)\langle A\xi, X\rangle$$

$$-\frac{(c-1)(3m+1)}{6}\eta(X)\langle\zeta, Y\rangle-\frac{(c-1)(m-1)}{6}\eta(Y)\langle\zeta, X\rangle$$

for any tangent vectors X, Y of M, where $\beta:=\langle A\xi, \xi\rangle$

Substituting $X=\xi$ into (3. 5), we see that

$$\begin{aligned}&(\mathrm{tr}\ A)A^2\xi-(tr\ A^2)A\xi\\ &=(\frac{c+3}{4}+\frac{c-1}{12}f^2)((\mathrm{tr}\ A)\xi-2mA\xi-(c-1)f\varphi A\zeta+\frac{c-1}{6}\{3\gamma+(5m+3)f^2\\ &-(3m+1)\}\zeta-\frac{c-1}{6}\{(3m+7)f^2-(3m+1)\}A\xi+\frac{c-1}{12}\{2m\beta-3(\mathrm{tr}\ A)(1-f^2)\}\xi,\end{aligned}\tag{3.6}$$

where $\gamma:=\langle A\zeta, \xi\rangle$. Substituting $Y=\xi$ into (3.5), we find

$$\begin{aligned}&(\mathrm{tr}\ A)A^2\xi-(tr\ A^2)A\xi\\ &=(\frac{c+3}{4}+\frac{c-1}{12}f^2)((\mathrm{tr}\ A)\xi-2mA\xi)-(c-1)f\varphi A\zeta\\ &+\frac{c-1}{6}\{3\gamma-(m+3)f^2-(m-1)\}\zeta\\ &-\frac{c-1}{6}\{(m+5)f^2-(m-1)\}A\xi+\frac{c-1}{12}\{2(3m+2)\beta-3(\mathrm{tr}\ A)(1-f^2)\}\xi.\end{aligned}\tag{3.7}$$

From (3.6) and (3.7), we obtain

$$(1-f^2)A\xi=(1-3f^2)\zeta+\beta\xi,\tag{3.8}$$

because of $c\neq 1$. From (3.4) and (3.8), it follows that

$$f\{(1-f^2)X-\langle\zeta, X\rangle\zeta-\eta(X)\xi\}=0\tag{3.9}$$

for any tangent vector X of M.

Let M_0 be a set consisting of points of M at which the function $1-f^2$ does not vanish. By virtue of Lemma 1, M_0 is a nonempty open set in M. There exists a nonzero tangent vector X at each point of M_0 such that $\langle X, \zeta\rangle=\langle X, \xi\rangle=0$, because $\dim M\geq 4$. Thus, from (3.9), we can see that the function f vanishes identically on M_0. Since M_0 is open and closed, we find $M_0=M$. Consequently the structure vector field $\tilde{\xi}$ of $\tilde{M}(c)$ tangent to M. Q. E. D.

THEOREM 2. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$, of dimension $2m+1$ (≥ 5). Then the structure tensor φ induced on M and the second fundamental tensor A derived from the unit normal commute each other, that is, $\varphi A=A\varphi$.*

PROOF. Combining (2.14) with Theorem 1 and using Lemma 2, we obtain

$$(\varphi A-A\varphi)X=\nabla_X\zeta-A\varphi X=\nabla_XA\xi-A\nabla_X\xi=(\nabla_XA)\xi=0$$

for any vector X tangent to M. Q. E. D.

LEMMA 4. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$,* *of dimension* $2m+1$ (≥ 5). *Then it follows that*

(3.10) $A\xi=\alpha\xi+\xi,$

(3.11) α *is constant on* M

and

(3.12) $A^2X=\alpha AX+\frac{c+3}{4}X-\frac{c-1}{4}\langle\xi, X\rangle\xi+\eta(X)\xi$

for any tangent vector X *of* M, *where* $\alpha:=\langle A\xi, \xi\rangle$.

PROOF. By Theorem 1 and Theorem 2, (3.5) reduces to

$$\begin{aligned}&(\mathrm{tr}\, A)\langle A^2, X, Y\rangle-(\mathrm{tr}\, A^2)\ \langle AX, Y\rangle\\ &=\frac{c+3}{4}\{(\mathrm{tr}\, A)\langle X, Y\rangle-2m\langle AX, Y\rangle\}\\ &+\frac{c-1}{2}\{\langle AX, Y\rangle-\langle A\xi, X\rangle\langle\xi, Y\rangle+\langle A\xi, Y\rangle\langle\xi, X\rangle-\langle\xi, X\rangle\eta(Y)\\ &+\langle\xi, Y\rangle\eta(X)\}-\frac{c-1}{4}(\mathrm{tr}\, A)\{\langle\xi, X\rangle\langle\xi, Y\rangle+\eta(X)\eta(Y)\}\end{aligned}\tag{3.13}$$

for any tangent vectors X, Y of M. Interchanging the role of X and Y in (3. 13), we see that

(3.14) $\langle\xi, X\rangle A\xi+\eta(X)\xi=\langle A\xi, X\rangle\xi+\langle\xi, X\rangle\xi$

for any tangent vector X of M. Substituting $X=\xi$ into this equation, we find (3.10). Combining (3.1) with (3.10) and using (2.14), it follows that

(3.15) $X\alpha=(\nabla_X h)(\xi, \xi)+2\langle A\xi, \nabla_X\xi\rangle=2\langle\alpha\xi+\xi, \varphi AX\rangle=0$

for any tangent vector X of M. That is , α is a constant on M. Differentiating (3.10) covariantly with any tangent vector X of M, and using (2. 14) and (3. 1), we obtain (3.12).

Q. E. D.

PROPOSITION 3. *Let* M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$, *of dimension* $2m+1$ (≥ 5). *If* $\alpha^2+c+3=0$ *on* M, *we have*

(3.16) $AX=\frac{\alpha}{2}(X+\langle\xi, X\rangle\xi-\eta(X)\xi)+\langle\xi, X\rangle\xi+\eta(X)\xi$

for any tangent vector X of M.

PROOF. In this case, by Lemma 4, we see that M has three constant princical curvatures $\frac{\alpha}{2}$, $\frac{\alpha+\sqrt{\alpha^2+4}}{2}$ and $\frac{\alpha-\sqrt{\alpha^2+4}}{2}$. Their multiplicities are $2m-2$, 1 and 1 respectively. Therefore we obtain (3. 16).

Q. E. D.

We have the following corollary of Proposition 3 :

COROLLARY 4. *Let* M *be a cyclic parallel hypersurface in a Sasakian space form* $\tilde{M}(-3)$ *of dimension* $2m+1$ (≥ 5). *If there exists a point* x *of* M *satisfying* $\alpha(x)=0$, *then* M *is totally contact geodesic.*

PROPOSITION 5. *Let* M *be a parallel hypersurface in a Sasakian space form* $\tilde{M}(1)$. *If the function* f *is a constant on* M, *then it follows that*

(3.17) $A\xi=\xi,$

(3.18) $A\zeta=\alpha\zeta+\xi$,
(3.19) $f=0$ (*i. e.*, $\tilde{\xi}$ *is tangent to* M),
(3.20) $\varphi A=A\varphi$
(3.21) α *is a constant on* M
and
(3.22) $A^2X=\alpha AX+X$
for any tangent vector X *of* M, *where* $\alpha:=\langle A\zeta, \zeta\rangle$.

PROOF. Since f is a constant on M, (3.17) is obvious. Substituting $Y=\zeta$ and $W=Z=\xi$ into (3.3) and using (3.17), we obtain

$$(1-f^2)A\zeta=\alpha\zeta+(1-f^2)\xi.$$

By Lemma 1, $1-f^2$ is a positive constant on M. Thus

$$A\zeta=\frac{\alpha}{1-f^2}\zeta+\xi. \tag{3.23}$$

Further, since M is parallel, we get

$$\begin{aligned} 0 &= (\nabla_X A)\xi \\ &= \nabla_X A\xi - A\nabla_X\xi \\ &= \nabla_X\zeta - A(\varphi X+fAX) \\ &= (\varphi A-A\varphi)X - f(A^2X+X) \end{aligned} \tag{3.24}$$

for any tangent vector X of M. Substituting $X=\xi$ into (3.24), we have

$$f\left(\frac{\alpha}{1-f^2}\zeta+2\xi\right)=0. \tag{3.25}$$

This shows (3.19), from which (3.18) and (3.20) are obtained. By a similar argument as the proof of Lemma 4, we obtain (3.21) and (3.22). Q. E. D.

§4. Hypersurfaces with $\varphi A=A\varphi$ in a Sasakian space form.

Let M be a hypersurface with $\varphi A=A\varphi$ in a Sasakian manifold $\tilde{M}$. We can see that $M_0:=\{x\in M \mid f^2(x)\neq 1\}$ is a nonempty open set in M, $\zeta\neq 0$ and $\xi\neq 0$ everywhere on M_0. From simple calculations, we get

$$\begin{aligned} &A\zeta=\alpha\zeta+\gamma\xi, \\ &A\xi=\gamma\zeta+\beta\xi, \\ &f(\alpha-\beta)=0 \text{ and } f\gamma=0 \text{ on } M_0, \end{aligned} \tag{4.1}$$

where $\alpha:=\dfrac{\langle A\zeta, \zeta\rangle}{1-f^2}$, $\beta:=\dfrac{\langle A\xi, \xi\rangle}{1-f^2}$ and $\gamma:=\dfrac{\langle A\zeta, \xi\rangle}{1-f^2}=\dfrac{\langle A\xi,\zeta\rangle}{1-f^2}$.

LEMMA 5. *Let* M *be a hypersurface with* $\varphi A=A\varphi$ *in a Sasakian manifold* $\tilde{M}$. *If* f *is a constant function on* M, *then it follows that*
(4.2) $A\xi=\zeta$, $f=0$, $A\zeta=\alpha\zeta+\xi$,
(4.3) $X\alpha=(\zeta\alpha)\langle\zeta, X\rangle$,

$$(4.4)\qquad A^2X=\alpha AX+\frac{c+3}{4}X-\frac{c-1}{4}(\langle\zeta, X\rangle\zeta+\eta(X)\xi),$$

$$(4.5)\qquad (\nabla_X A)\xi=0$$

and

$$(4.6)\qquad (\nabla_X A)\zeta=-\frac{c-1}{4}\varphi X+(\zeta\alpha)\langle\zeta, X\rangle\zeta$$

for any tangent vector X of M.

PROOF. From (2.14) and (4.1), we have (4.2) everywhere on M. Differentiating $A\zeta$ covariantly with any tangent vector X of M and using (2.14) and (4.2), we obtain

$$(4.7)\qquad (\nabla_X A)\zeta=-\varphi A^2X+\alpha\varphi AX+\varphi X+(X\alpha)\zeta.$$

Using the equation of Codazzi (2.17) and (4.7), we have

$$(4.8)\qquad (X\alpha)\zeta-(\nabla_\zeta A)X=\varphi A^2X-\alpha\varphi AX-\frac{c+3}{4}\varphi X,$$

from which

$$(4.9)\qquad (X\alpha)\langle\zeta, Y\rangle-(Y\alpha)\langle\zeta, X\rangle=2\{\langle\varphi A^2X, Y\rangle-\alpha\langle\varphi AX, Y\rangle-\frac{c+3}{4}\langle\varphi X, Y\rangle\}$$

for any tangent vectors X, Y of M. Substituting $Y=\zeta$ into (4.9), we find (4.3). Substituting (4.3) into (4.9), we have

$$(4.10)\qquad \varphi A^2X-\alpha\varphi AX-\frac{c+3}{4}\varphi X=0$$

for any tangent vector X of M. From (2.14) and (4.10), we obtain (4.4). (4.5) is obvious. From (4.3), (4.4) and (4.7), we have (4.6). Q. E. D.

PROPOSITION 6. *Let M be a hypersurface with $\varphi A=A\varphi$ in a Sasakian space form $\tilde{M}(c)$, $c\neq-3$, of dimension $2m+1(\geq 5)$. If f is a constant function on M, then M is cyclic parallel.*

PROOF. Differentiating (4.4) covariantly with ζ, we find

$$(4.11)\qquad (\zeta\alpha)(AX-\langle\alpha\zeta+\xi, X\rangle\zeta-\langle\zeta, X\rangle\xi)=0$$

for any tangent vector X of M.

Let M_1 be a set consisting points of M at which the functoin $\zeta\alpha$ does not vanish, and suppose that M_1 is not empty. From (4. 11), it follows that

$$AX=\langle\alpha\zeta+\xi, X\rangle\zeta+\langle\zeta, X\rangle\xi$$

for any tangent vector X of M_1. Combining this with (4. 4), we find

$$(c+3)(X-\langle\zeta, X\rangle\zeta-\eta(X)\xi)=0$$

for any tangent vector X of M_1. Thus the assumption of M_1 produces a contradiction because $c\neq-3$ and dim $M\geq 4$. Accordingly we obtain

$$(4.12)\qquad \zeta\alpha=0 \text{ (everywhere on } M).$$

Therefore, from this and (4.3) we see that α is a constant on M. Using this fact, (4.6) reduces to

$$(4.13)\qquad (\nabla_X A)\zeta=-\frac{c-1}{4}\varphi X$$

for any tangent vector X of M.

Differentiating (4.4) covariantly with any tangent vector Y of M, we obtain

$$(4.14)\quad \begin{aligned}(\nabla_Y A)AX+A(\nabla_Y A)X=\alpha(\nabla_Y A)X-\frac{c-1}{4}\{&\langle\varphi AY, X\rangle+\langle\xi, X\rangle\varphi AY\\&+\langle\varphi Y, X\rangle\xi+\eta(X)\varphi Y\}.\end{aligned}$$

Interchanging the role of X and Y in the above equation and combing these equations with the equation of Codazzi (2.17), we get

$$(4.15)\quad \begin{aligned}&(\nabla_X A)AY-(\nabla_Y A)AX\\&=\frac{c-1}{4}\{\langle\alpha\xi+\xi, X\rangle\varphi Y-\langle\alpha\xi+\xi, Y\rangle\varphi X-2\langle\varphi AX, Y\rangle\xi\},\end{aligned}$$

from which

$$(4.16)\quad \begin{aligned}&(\nabla_X A)AY-A(\nabla_X A)Y\\&=\frac{c-1}{4}\{\langle\varphi X, Y\rangle(\alpha\xi+\xi)-\langle\alpha\xi+\xi, Y\rangle\varphi X-\langle\varphi AX, Y\rangle\xi+\langle\xi, Y\rangle\varphi AX\}.\end{aligned}$$

From (4.14) and (4.16), we have

$$(4.17)\quad \begin{aligned}2(\nabla_X A)AY=\alpha(\nabla_X A)Y+\frac{c-1}{4}\{&\alpha\langle\varphi X, Y\rangle\xi-\alpha\langle\xi, Y\rangle\varphi X\\&-2\langle\varphi AX, Y\rangle\xi-2\eta(Y)\varphi X\}\end{aligned}$$

for any tangent vectors X, Y of M. Combining this with (4.4), it follows that

$$(4.18)\quad (\alpha^2+c+3)\{(\nabla_X A)Y+\frac{c-1}{4}\{\langle\varphi X, Y\rangle\xi+\langle\xi, Y\rangle\varphi X)\}=0$$

for any tangent vectors X, Y of M. Thus M is cyclic parallel provided that $\alpha^2+c+3\neq 0$.

Next, assuming that $\alpha^2+c+3=0$, (4. 2) and (4. 4) show that M has three constant principal curvatures $\frac{\alpha}{2}$, $\frac{\alpha+\sqrt{\alpha^2+4}}{2}$, and $\frac{\alpha-\sqrt{\alpha^2+4}}{2}$. Their multiplicities are $2m-2$, 1, and 1 respectively. This gives

$$(4.19)\quad AX=\frac{\alpha}{2}(X+\langle\xi, X\rangle\xi-\eta(X)\xi)+\langle\xi, X\rangle\xi+\eta(X)\xi$$

for any tangent vector X of M. Differentiating this covariantly, we find

$$(4.20)\quad \begin{aligned}(\nabla_X A)Y&=(\frac{\alpha^2}{4}+1)(\langle\varphi X, Y\rangle\xi+\langle\xi, Y\rangle\varphi X)\\&=-\frac{c-1}{4}(\langle\varphi X, Y\rangle\xi+\langle\xi, Y\rangle\varphi X)\end{aligned}$$

for any tangent vectors X, Y of M. Thus M is cyclic parallel because of Lemma 2.

Q. E. D.

From the proof of Proposition 6, we have the following

REMARK. *Let M be a hypersurface with $\varphi A=A\varphi$ in a Ssasakian space form $\tilde{M}(-3)$. If f is a constant and $\xi\alpha=0$ everywhere on M, then M is cyclic parallel.*

Let M be a hypersurface with $\varphi A=A\varphi$ in a Ssasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$ and M_2 a set consisting of points of M at which $0<f^2<1$. Assume that f is a nonconstant function on M, then M_2 is a nonempty open set in M because of Lemma 1. Thus M_2 is a hypersurface (not necessarily connected) in $\tilde{M}(c)$, and we have

THEOREM 7. *Let M be a hypersurface with $\varphi A=A\varphi$ in a Ssasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$. If f is a nonconstant function on M, then M_2 is a totally umbilical hypersurface. Therefore $c=1$ and M_2 is parallel (and the mean curvature on each connected component of M_2 is a constant).*

PROOF. For this case,

(4.21) $\quad A\zeta=\alpha\zeta$

and

(4.22) $\quad A\xi=\alpha\xi$

on M_2, where $\alpha:=\frac{\langle A\zeta,\zeta\rangle}{1-f^2}=\frac{\langle A\xi,\xi\rangle}{1-f^2}$. Differentiating (4. 21) and (4. 22) covariantly with any tangent vector X of M_2, and using (2. 14), we obtain

(4.23) $\quad (\nabla_X A)\zeta=(X\alpha)\zeta+\alpha(\varphi AX-f\,X)-\varphi A^2X+fAX$

and

(4.24) $\quad (\nabla_X A)\xi=(X\alpha)\xi+\alpha(\varphi X+fAX)-\varphi AX-fA^2X.$

Combining (5.24) with the equation of Codazzi (2.17), it follows that

$$(4.25)\quad \begin{aligned}(\nabla_\xi A)X=&(X\alpha)\xi+\alpha(\varphi X+fAX)-\varphi AX-fA^2X\\ &-\frac{c-1}{4}f\{(1-f^2)X+3\langle\zeta,X\rangle\zeta-\eta(X)\xi\}\end{aligned}$$

for any tangent vector X of M_2. Taking the inner product with ξ, we have

(4.26) $\quad (1-f^2)(X\alpha)=(\xi\alpha)\eta(X)$

for any tangent vector X of M_2. Substituting $X=\zeta$ into (4. 25), we find

(4.27) $\quad \zeta\alpha=0$ and $\xi\alpha=-(c-1)f(1-f^2)$ on M_2.

From (4.26) and(4.27), we get

(4.28) $\quad X\alpha=-(c-1)f\,\eta(X)$

for any tangent vector X of M_2. Using this equation, (4.25) reduces to

$$(4.29)\quad (\nabla_\xi A)X=\alpha(\varphi X+fAX)-\varphi AX-fA^2X-\frac{c-1}{4}f\{(1-f^2)X+3\langle\zeta,X\rangle\zeta+3\eta(X)\xi\}$$

for any vector X tangent to M_2. Taking the inner product with any tangent vector Y of M_2 and interchanging the role of X and Y, we find

(4.30) $\quad AX=\alpha X$

for any tangent vector X of M_2. Thus M_2 is a totally umbilical hypersurface with mean curvature $\rho=\alpha$. Therefore $c=1$, M_2 is a parallel hypersurface and the mean curvature a constant on each connected component of M_2, by virture of Lemma 3. Q. E. D.

PROPOSITION 8. *Let M be a hypersurface with $\varphi A=A\varphi$ in a Ssasakian space form $\tilde{M}(c)$*

of dimension $2m+1(\geq 5)$. *If* f *is a nonconstant function on* M, *then* M *is parallel and* $c=1$.

PROOF. By Lemma 2, $M_2 := \{x \in M \mid 0 < f^2(x) < 1\}$ is a parallel hypersurface in $\tilde{M}(c)$ and $c=1$. We put $M_3 := \{x \in M \mid \|\nabla A\|(x) \neq 0\}$. Suppose that M_3 is not empty, then since $M_3 \subset M - M_2$, it follows that

$$f(x)=0 \quad \text{or} \quad f^2(x)=1 \quad \text{for any } x \in M_3.$$

If there exists some point x of M_3 satisfying $f^2(x)=1$, we see that $U \cap M_2 \neq \phi$ for any neighborhood U of x in M, i. e., x is an accumulation point of M_2, because of Lemma 1. Thus we have $\|\nabla A\|(x)=0$. This is a contradiction.
Therefore we obtain $M_3 \subset \{x \in M \mid f(x)=0\}$. In this case, by Proposition 6, M_3 is cyclic parallel. Since $c=1$, M_3 is parallel. Therefore the assumption of M_3 produces a contradiction. Accordingly M is parallel. Q. E. D.

Proposition 6 and Proposition 8 assert the following

THEOREM 9. *Let* M *be a hypersurface with* $\varphi A = A\varphi$ *in a Sasakian space form* $\tilde{M}(c)$, $c \neq -3$, *of dimension* $2m+1$ (≥ 5). *Then* M *is a cyclic parallel.*

THEOREM 10. *Let* M *be a complete hypersurface with* $\varphi A = A\varphi$ *in a Sasakian space form* $\tilde{M}(c)$ *of dimension* $2m+1(\geq 5)$. *If* f *does not vanish everywhere on* M, *then* M *is a totally umbilical hypersurface with constant mean curvature, isometric to an ordinary sphere, and* $c=1$.

PROOF. By Lemma 5, f is a nonconstant function on M. Using Lemma 1, we see that any point of M is an accumulation point of M_2. Thus M is a totally umbilical hypersurface with constant mean curvature $\rho=\alpha$, by virtue of Theorem 7. In this case, we have

$$(\nabla\nabla f)(X, Y) := (\nabla_X df)Y = -(1+\alpha^2)f\langle X, Y\rangle$$

for any tangent vector fields X, Y on M. By virtue of Obata's theorem [6], we see that M is isometric to an ordinary sphere of radius $\sqrt{1+\alpha^2}$. Q. E. D.

§5. Cyclic parallel and totally contact umbilical hypersurfaces.

Let M be a totally contact umbilical hypersurface in a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$ and ρ the mean curvature of M in $\tilde{M}$. Then the second fundamental form h has the following form :

$$\begin{aligned} & h(\xi, X) = \langle \zeta, X\rangle, \\ (5.1) \qquad & h(X, Y) = \alpha\{\langle X, Y\rangle - \eta(X)\eta(Y)\} + \eta(X)h(\xi, Y) + \eta(Y)h(\xi, X) \\ & \qquad = \alpha\{\langle X, Y\rangle - \eta(X)\eta(Y)\} + \eta(X)\langle\zeta, Y\rangle + \eta(Y)\langle\zeta, X\rangle, \end{aligned}$$

where $\alpha := \frac{2m}{2m-1}\rho$. (5. 1) is equivalent to

$$(5.2) \qquad A\xi = \zeta, \quad A\zeta = \alpha\zeta + \xi \quad \text{and} \quad \varphi A = \alpha\varphi \ (= A\varphi).$$

PROPOSITION 11. *Let M be a totally contact umbilical hypersurface in a Sasakian manifold $\tilde{M}$. Then M is cyclic parallel if and only if the mean curvature ρ of M in $\tilde{M}$ is a constant. In this casa, we have*

$$(\nabla_X h)(Y, Z) = \langle \varphi X, Y\rangle\langle \xi, Z\rangle + \langle \varphi X, Z\rangle\langle \xi, Y\rangle.$$

PROOF. Differentiating (5. 1) covariantly and making use of (2.14), we find

$$(5.3)\quad \begin{aligned}(\nabla_X h)(Y, Z) = &-\alpha\{\langle \varphi X, Y\rangle \eta(Z) + \langle \varphi X, Z\rangle \eta(Y)\} + \langle \varphi X, Y\rangle\langle \xi, Z\rangle + \langle \varphi X, Z\rangle\langle \xi, Y\rangle \\ &+ \langle \varphi AX, Y\rangle \eta(Z) + \langle \varphi AX, Z\rangle \eta(Y) + (X\alpha)(\langle Y, Z\rangle - \eta(Y)\eta(Z)). \\ &= \langle \varphi X, Z\rangle\langle \xi, Z\rangle + \langle \varphi X, Z\rangle\langle \xi, Y\rangle + (X\alpha)(\langle Y, Z\rangle - \eta(Y)\eta(Z)).\end{aligned}$$

From this, we get

$$(5.4)\quad \begin{aligned}&(\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X) + (\nabla_Z h)(X, Y) \\ &= (X\alpha)(\langle Y, Z\rangle - \eta(Y)\eta(Z)) + (Y\alpha)(\langle Z, X\rangle - \eta(Z)\eta(X)) \\ &\quad + (Z\alpha)(\langle X, Y\rangle - \eta(X)\eta(Y)).\end{aligned}$$

If M is cyclic parallel, we obtain

$$(5.5)\quad X\alpha = \xi\alpha = 0 \quad (X \perp \xi).$$

Thus we see that α is a constant, i. e., ρ is a constant.

Conversely, assume that ρ is a constant. From (5. 4), we see that

$$(\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X) + (\nabla_Z h)(X, Y) = 0,$$

that is, M is cyclic parallel. Q. E. D.

PROSITION 12. *Let M be a totally contact umbilical hypersurface in a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$. Then $c=-3$ and M is cyclic parpllel.*

PROOF. Since M is totally contact umbilical, we have

$$(5.6)\quad (\nabla_X A)Y = (X\alpha)(Y - \eta(Y)\xi) + \langle \varphi X, Y\rangle\xi + \langle \xi, Y\rangle \varphi X$$

for any tangent vectors X, Y of M, where $\alpha := \frac{2m}{2m-1}\rho$. Using the equation of Codazzi (2.17), we obtain

$$(5.7)\quad \begin{aligned}&(X\alpha)(Y - \eta(Y)\xi) - (Y\alpha)(X - \eta)(X)\xi) \\ &= \frac{c+3}{4}(\langle \xi, X\rangle \varphi Y - \langle \xi, Y\rangle \varphi X + 2\langle X, \varphi Y\rangle \xi)\end{aligned}$$

for any tangent vectors X, Y of M. Since dim $M \geq 4$, there exists a nonzero tangent vector X such that $\langle \xi, X\rangle = \eta(X) = 0$. Therefore, substituting $Y = \xi$ into (5.7) and taking the inner product with φX, we have $c=-3$. From this and (5.7), we have

$$(5.8)\quad (X\rho)(\langle Y, Z\rangle - \eta(Y)\eta(Z)) = (Y\rho)(\langle X, Z\rangle - \eta(X)\eta(Z))$$

for any vectors X, Y, Z tangent to M. Sustituting $Y = Z = \xi$ into (5.8), we see that ρ is a constant. By virtue of Proposition 11, M is cyclic parallel. Q. E. D.

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