Cyclic Parallel Hypersurfaces in a Sasakian Space Form

メタデータ	言語: English 出版者: 北海道教育大学 公開日: 2012-11-07 キーワード: 作成者: 長谷川, 和泉 メールアドレス: 所属:
URL	https://doi.org/10.32150/00003801

北海道教育大学紀要（第2部A）第41巻　第2号
Journal of Hokkaido University of Education (Section II A) Vol. 41, No. 2

平成3年3月
March, 1991

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佐々木空間形の巡回平行超曲面

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Abstract

We investigate cyclic parallel hypersurfaces in a Sasakian space form and prove some theorems.

§1. Introduction.

Let M be a submanifold in a Riemannian manifold $\tilde{M}$. If the second fundamental form σ of M in $\tilde{M}$ is cyclic parallel, that is,

$$(\bar{\nabla}_X \sigma)(Y, Z) + (\bar{\nabla}_Y \sigma)(Z, X) + (\bar{\nabla}_Z \sigma)(X, Y) = 0$$

for arbitrary vectors X, Y, Z tangent to M, then M is said to be cyclic parallel.

Recently, U-H. Ki [5] has proved that a real hypersurface M in a real $2m(\geq 4)$-dimensional complex space form $\tilde{M}(c)$ with nonzero constant holomorphic sectional curvature c is cyclic parallel if and only if $\varphi A = A\varphi$, where φ denotes the structure tensor induced on M by almost complex structure of $\tilde{M}(c)$ and A the second fundamental tensor derived from the unit normal.

In this paper, we investigate cyclic parallel hypersurfaces in a Sasakian space form and prove the following theorems :

THEOREM 1. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c \neq 1$, of dimension $2m+1(\geq 5)$. Then the structure vector field ξ of $\tilde{M}(c)$ is tangent to M.*

THEOREM 2. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$, of dimension $2m+1(\geq 5)$. Then the structure tensor φ induced on M and the second fundamental tensor A derived from the unit normal commute into each other, that is, $\varphi A=A\varphi$.*

THEOREM 9. *Let M be a hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq -3$, of dimension $2m+1(\geq 5)$. If the structure tensor φ induced on M and the second fundamental tensor A derived from the unit normal commute into each other, then M is cyclic parallel.*

THEOREM 10. *Let M be a complete hypersurface with $\varphi A=A\varphi$ in a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$, where φ is the structure tensor induced on M, and A the second fundamental tensor derived from the unit normal. If the structure vector field $\tilde{\xi}$ of $\tilde{M}(c)$ is not tangent to M evrywhere on M, then M is a totally umbilical hypersurface with constant mean curvature, isometric to an ordinary sphere, and $c=1$.*

Throughout this paper, we assume that all objects under consideration are differentiable of class C^∞ and that all manifolds are connected unless otherwise stated.

The author would like to express his gratitude to Professors T. Nagai and H. Kôjyô for their valuable suggestions and criticism.

§2. Preliminaries.

This section introduces some definitions and the fundamental properties used throughout the paper.

(1) Let M be a $(2m+1)$-dimensional Sasakian manifold. We denote by $(\varphi, \xi, \eta, \langle\ ,\ \rangle)$ The Sasakian structure of M, where φ is a tensor field of type (1, 1), ξ a vector field, η a 1-form and $\langle\ ,\ \rangle$ a Riemannian metric. The structure tensors satisfy the following equations :

$$\begin{aligned} &\varphi^2X=-X+\eta(X)\xi, \quad \varphi\xi=0, \quad \eta(\varphi X)=0, \quad \eta(\xi)=1, \\ &\langle\varphi X, Y\rangle+\langle X, \varphi Y\rangle=0, \quad \eta(X)=\langle\xi, X\rangle, \\ &\nabla_X\xi=\varphi X, \quad (\nabla_X\varphi)Y=\eta(Y)X-\langle X, Y\rangle\xi \end{aligned} \tag{2.1}$$

for any vector fields X, Y tangent to M, where ∇ denotes the Riemannian connection of M.

The Riemannian curvature tensor R of the Sasakian manifold M, defined by $R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$, satisfies

$$\begin{aligned} &R(\varphi X, \varphi Y)Z=R(X, Y)Z-\langle Y, Z\rangle X+\langle X, Z\rangle Y-\langle Y, \varphi Z\rangle\varphi X+\langle X, \varphi Z\rangle\varphi Y, \\ &R(X, \varphi Y)Z=-R(\varphi X, Y)Z-\langle Y, \varphi Z\rangle X+\langle X, \varphi Z\rangle Y+\langle Y, Z\rangle\varphi X-\langle X, Z\rangle\varphi Y \\ &\text{and} \\ &R(X, Y)\xi=\eta(Y)X-\eta(X)Y. \end{aligned} \tag{2.2}$$

A Sasakian manifold M is called a Sasakian space form if M is of constant φ-holomorphic sectional curvature. The Riemannian curvature tensor R of the Sasakian space form $M(c)$ of constant φ-holomorphic sectional curvature c takes the following form :

$$\begin{aligned} R(X, Y)Z=&\frac{c+3}{4}\{\langle Y, Z\rangle X-\langle X, Z\rangle Y\}+\frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X \\ &+\langle X, Z\rangle\eta(Y)\xi-\langle Y, Z\rangle\eta(X)\xi+\langle X, \varphi Z\rangle\varphi Y \\ &-\langle Y, \varphi Z\rangle\varphi X+2\langle X, \varphi Y\rangle\varphi Z\}. \end{aligned} \tag{2.3}$$

(2) Let $\tilde{M}$ be a Riemannian manifold and M a Riemannian manifold isometrically immersed in $\tilde{M}$. Then M is called a submanifold in $\tilde{M}$. Particularly, M is called a hypersurface in $\tilde{M}$ if codim $M=1$. The Riemannian meteic on $\tilde{M}$ as well as the induced metric on M is denoted by $\langle\ ,\ \rangle$. Let ∇ and $\tilde{\nabla}$ be the Riemannian connections on M and $\tilde{M}$, respectively. Then the Gauss and Weingarten formulas are given by

(2.4) $\tilde{\nabla}_X Y=\nabla_X Y+\sigma(X, Y)$

and

(2.5) $\tilde{\nabla}_X N=-A_N X+\nabla^{\perp}_X N$

for any vector fields X, Y tangent to M and N normal to M, where σ denotes the second fundamental form, A_N the second fundamental tensor at N and $\nabla^{\perp}$ the linear connection induced in the normal bundle $T^{\perp}M$, called the normal connection. The second fundamental tensor A_N is related to the second fundamental form σ by

(2.6) $\langle A_N,X, Y\rangle=\langle\sigma(X, Y), N\rangle$.

Denoting the Riemannian curvature tensors of M and $\tilde{M}$ by R and $\tilde{R}$ respectively, the equations of Gauss and Codazzi are given by

(2.7) $\langle W, R(X, Y)Z\rangle=\langle W, \tilde{R}(X, Y)Z\rangle+\langle\sigma(W, X), \sigma(Y, Z)\rangle-\langle\sigma(W, Y), \sigma(X, Z)\rangle$

and

(2.8) $\langle\tilde{R}(X, Y)Z, N\rangle=\langle(\bar{\nabla}_X\sigma)(Y, Z), N\rangle-(\bar{\nabla}_Y\sigma)(X, Z), N\rangle$

for any vectors W, X, Y, Z tangent to M and N normal to M, where the first covariant differentiation $\bar{\nabla}\sigma$ of σ is defined by

(2.9) $(\bar{\nabla}_X\sigma)(Y, Z)=\nabla^{\perp}_X\sigma(Y, Z)-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z)$.

A submanifold M is called a parallel submanifold if σ is parallel, i. e., $\bar{\nabla}\sigma=0$ identically. A submanifold M is called a cyclic parallel submanifold if the cyclic sum of $(\bar{\nabla}_X\sigma)(Y, Z)$ vanishes identically, i. e.,

(2.10) $(\bar{\nabla}_X\sigma)(Y, Z)+(\bar{\nabla}_Y\sigma)(Z, X)+(\bar{\nabla}_Z\sigma)(X, Y)=0$

for any vectors X, Y, Z tangent to M. It is easily seen that condition (2.10) is equivalent to

(2.11) $(\bar{\nabla}_X\sigma)(X, X)=0$ for all $X\in TM$.

It was proved in [2] that any geodesic hypersphere in a complex space form with non-zero constant holomorphic sectional curvature is cyclic parallel and not parallel.

Let M be submanifold in a Sasakian manifold $\tilde{M}$ with structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \langle\ ,\ \rangle)$. M is said to be anti-invariant if

(2.12) $\tilde{\varphi}(T_xM)\subset T_xM$ for each $x\in M$.

We have the following well-known lemma :

LEMMA 1(e. g., see [9]). *Let M be an n-dimensional submanifold in a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$. If the structure vector field $\tilde{\xi}$ is normal to M, then M is anti*

-invariant, and $m \geq n$.

(3) Let M be a hypersurface in a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$ with Sasakian structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}\ \langle\ ,\ \rangle)$. A unit normal ε to M may then be chosen. For this unit normal ε, we put

$$\begin{aligned} &f := \tilde{\eta}(\varepsilon),\ \zeta := -\varphi\varepsilon,\ \ \xi := \tilde{\xi} - f\varepsilon,\ \varphi X := \tilde{\varphi}X - \langle \zeta, X \rangle \varepsilon \\ &AX := A_\varepsilon X \text{ and } h\langle X, Y \rangle := \langle AX, Y \rangle = \langle \sigma(X, Y), \varepsilon \rangle \end{aligned} \tag{2. 13}$$

for any vectors X, Y tangent to M. By the properties of the Sasakian structure, the following relations are given :

$$\begin{aligned} &\langle \zeta, \xi \rangle = 0, \quad \| \zeta \|^2 = \| \xi \|^2 = 1 - f^2, \quad \varphi\zeta = -f\xi, \quad \varphi\xi = f\zeta, \\ &\varphi^2 X = -X + \langle \zeta, X \rangle \zeta + \eta(X)\xi, \quad \langle \varphi X, Y \rangle = -\langle X, \varphi Y \rangle \\ &\nabla_X \zeta = \varphi AX - f X, \quad \nabla_X \xi = \varphi X + fAX, \quad Xf = \langle \zeta - A\xi, X \rangle, \\ &(\nabla_X \varphi) Y = \langle \zeta, Y \rangle AX - \langle AX, Y \rangle \zeta + \eta(Y) X - \langle X, Y \rangle \xi \end{aligned} \tag{2.14}$$

for any vectors X, Y tangent to M, where $\eta(X) := \langle \xi, X \rangle$.

A scalar function $\rho := \frac{1}{2m}$trace A is called a mean curvature of M in $\tilde{M}$. M is said to be totally umbilical if $AX = \rho X$ for any vector X tangent to M. Particularly, M is said to be totally geodesic, if $AX = 0$ for any vector X tangent to M. If the structure vector field $\tilde{\xi}$ is tangent to M and

$$AX = \frac{2m}{2m-1}\rho(X - \eta(X)\xi) + \eta(X)A\xi + \langle A\xi, X \rangle \xi \tag{2.15}$$

for any vector X tangent to M, then M is said to be totally contact umbilical. When a totally contact umbilical hypersurface M has vanishing mean curvature, then M is said to be totally contact geodesic.

In the following, the ambient Sasakian manifold is assumed to be a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1$. Then the equations of Gauss and Codazzi for M in $\tilde{M}(c)$ are respectively rewritten as :

$$\begin{aligned} R(X, Y)Z = \frac{c+3}{4}(\langle Y, Z \rangle X - \langle X, Z \rangle Y) + \frac{c-1}{4}\{&\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \langle X, Z \rangle \eta(Y)\xi - \langle Y, Z \rangle \eta(X)\xi + \langle X, \varphi Z \rangle \varphi Y - \langle Y, \varphi Z \rangle \varphi X \\ &+ 2\langle X, \varphi Y \rangle \varphi Z\} + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY. \end{aligned} \tag{2.16}$$

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = \frac{c-1}{4}\{&f(\eta(Y)X - \eta(X)Y) + \langle \zeta, X \rangle \varphi Y - \langle \zeta, Y \rangle \varphi X \\ &+ 2\langle X, \varphi Y \rangle \zeta\}, \end{aligned} \tag{2.17}$$

i. e.,

$$\begin{aligned} (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = \frac{c-1}{4}\{&f(\eta(Y)\langle X, Z \rangle - \eta(X)\langle Y, Z \rangle) \\ &+ \langle \zeta, Y \rangle \langle X, \varphi Z \rangle - \langle \zeta, X \rangle \langle Y, \varphi Z \rangle + 2\langle \zeta, Z \rangle \langle X, \varphi Y \rangle\}. \end{aligned}$$

LEMMA 2. *Let* M *be a hypersurface in a Sasakian space form* $\tilde{M}(c)$. *Then* M *is cyclic parallel if and only if*

$$(2.18)\quad \begin{aligned}(\nabla_X h)(Y, Z) &= \frac{c-1}{4}(\langle\xi, Y\rangle\langle X, \varphi Z\rangle + \langle\xi, Z\rangle\langle X, \varphi Y\rangle)\\ &+ \frac{c-1}{12} f(\langle X, Y\rangle\eta(Z) + \langle X, Z\rangle\eta(Y) - 2\langle Y, Z\rangle\eta(X))\end{aligned}$$

for any vectors X, Y, Z tangent to M.

The proof for Lemma 2 is simple and has been omitted.

Lemma 2 simply leads to the following

REMARK. *If there exists a parallel hypersurface in a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$, then $c=1$.*

It is known that a totally umbilical hypersurface M in a Riemannian manifold $\tilde{M}$ is parallel if and only if the mean curvature ρ of M in $\tilde{M}$ is a constant. For Sasakian geometry, we have

LEMMA 3 [4, 8]. *Let M be a totally umbilical hypersurface in a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$. Then $c=1$ and M is parallel.*

§3. Cyclic parallel hypersurface in a Sasakian space form.

Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$. By Lemma 2, we have

$$(3.1)\quad \begin{aligned}&(\nabla h)(X, Y, Z) := (\nabla_X h)(Y, Z)\\ &= \frac{c-1}{4}(\langle\xi, Y\rangle\langle X, \varphi Z\rangle + \langle\xi, Z\rangle\langle X, \varphi Y\rangle)\\ &+ \frac{c-1}{12} f(\langle X, Y\rangle\eta(Z) + \langle X, Z\rangle\eta(Y) - 2\langle Y, Z\rangle\eta(X))\end{aligned}$$

for any vectors X, Y, Z tangent to M. By differentiating this covariantly along M and making use of (2. 14), we find

$$(3.2)\quad \begin{aligned}&(\nabla\nabla h)(W, X, Y, Z) := (\nabla_W(\nabla h))(X, Y, Z)\\ &= \frac{c-1}{4}\{(\langle AW, X\rangle\langle\xi, Y\rangle - \langle AW, Y\rangle\langle\xi, X\rangle + \langle W, X\rangle\eta(Y) - \langle W, Y\rangle\eta(X))\langle\xi, Z\rangle\\ &+ (\langle AW, X\rangle\langle\xi, Z\rangle - \langle AW, Z\rangle\langle\xi, X\rangle + \langle W, X\rangle\eta(Z) - \langle W, Z\rangle\eta(X))\langle\xi, Y\rangle\\ &+ \langle X, \varphi Y\rangle(\langle\varphi AW, Z\rangle - f\langle W, Z\rangle) + \langle X, \varphi Z\rangle(\langle\varphi AW, Y\rangle - f\langle W, Y\rangle)\}\\ &+ \frac{c-1}{12}\langle\xi - A\xi, W\rangle(\langle X, Y\rangle\eta(Z) + \langle X, Z\rangle\eta(Y) - 2\langle Y, Z\rangle\eta(X))\\ &+ \frac{c-1}{12} f\{\langle X, Y\rangle(\langle\varphi W, Z\rangle + f\langle AW, Z\rangle) + \langle X, Z\rangle(\langle\varphi W, Y\rangle + f\langle AW, Y\rangle)\\ &- 2\langle Y, Z\rangle(\langle\varphi W, X\rangle + f\langle AW, X\rangle)\}\end{aligned}$$

for any vectors W, X, Y, Z tangent to M.

Substituting this and (2. 16) into the Ricci formula given by

$$(\nabla\nabla h)(W, X, Y, Z)-(\nabla\nabla h)(X, W, Y, Z)=-\langle R(W, X)Y, AZ\rangle-\langle R(W, X)Z, AY\rangle,$$

it follows that

$$\begin{aligned}
&\langle AW, Y\rangle\langle A^2X, Z\rangle+\langle AW, Z\rangle\langle A^2X, Y\rangle-\langle AX, Y\rangle\langle A^2W, Z\rangle-\langle AX, Z\rangle\langle A^2W, Y\rangle\\
&=(\frac{c+3}{4}+\frac{c-1}{12}f^2)(\langle AW, Y\rangle\langle X, Z\rangle+\langle AW, Z\rangle\langle X, Y\rangle-\langle AX, Y\rangle\langle W, Z\rangle\\
&\qquad\qquad-\langle AX, Z\rangle\langle W, Y\rangle)\\
&\quad-\frac{c-1}{4}\{(\langle AW, Y\rangle\langle\zeta, X\rangle-\langle AX, Y\rangle\langle\zeta, W\rangle+\langle W, Y\rangle\eta(X)-\langle X, Y\rangle\eta(W))\langle\zeta, Z\rangle\\
&\qquad+\langle AW, Z\rangle\langle\zeta, X\rangle-\langle AX, Z\rangle\langle\zeta, W\rangle+\langle W, Z\rangle\eta(X)-\langle X, Z\rangle\eta(W))\langle\zeta, Y\rangle\\
&\qquad-\langle(\varphi A-A\varphi)W, Y\rangle\langle X, \varphi Z\rangle+\langle(\varphi A-A\varphi)X, Y\rangle\langle W, \varphi Z\rangle\\
&\qquad-\langle(\varphi A-A\varphi)W, Z\rangle\langle X, \varphi Y\rangle+\langle(\varphi A-A\varphi)X, Z\rangle\langle W, \varphi Y\rangle\\
&\qquad+2\langle(\varphi A-A\varphi)Y, Z\rangle\langle W, \varphi X\rangle\\
&\qquad+\langle AW, Y\rangle\eta(X)\eta(Z)-\langle AX, Y\rangle\eta(W)\eta(Z)+\langle AW, Z\rangle\eta(X)\eta(Y)\\
&\qquad-\langle AX, Z\rangle\eta(W)\eta(Y)+\eta(W)\langle X, Y\rangle\langle A\xi, Z\rangle-\eta(X)\langle W, Y\rangle\langle A\xi, Z\rangle\\
&\qquad+\eta(W)\langle X, Z\rangle\langle A\xi, Y\rangle-\eta(X)\langle W, Z\rangle\langle A\xi, Y\rangle\}\\
&\quad+\frac{c-1}{6}f(\langle W, \varphi Y\rangle\langle X, Z\rangle-\langle X, \varphi Y\rangle\langle W, Z\rangle+\langle W, \varphi Z\rangle\langle X, Y\rangle\\
&\qquad-\langle X, \varphi Z\rangle\langle W, Y\rangle+2\langle W, \varphi X\rangle\langle Y, Z\rangle)\\
&\quad+\frac{c-1}{12}\langle\zeta-A\xi, W\rangle\ (\eta(Y)\langle X, Z\rangle+\eta(Z)\langle X, Y\rangle-2\eta(X)\langle Y, Z\rangle)\\
&\quad-\frac{c-1}{12}\langle\zeta-A\xi, X\rangle\ (\eta(Y)\langle W, Z\rangle+\eta(Z)\langle W, Y\rangle-2\eta(W)\langle Y, Z\rangle)
\end{aligned}\tag{3. 3}$$

for any vectors W, X, Y, Z tangent to M.

THEOREM 1. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$, of dimension $2m+1$ (≥ 5). Then the structure vector field $\tilde{\xi}$ of $\tilde{M}(c)$ is tangent to M.*

PROOF. Let $\{E_1,\cdots\cdots, E_{2m}\}$ be an orthonormal basis of T_xM for any point $x\in M$. Substituting $Y=Z=E_i$ into (3.3) and summing up i from 1 to $2m$, we have

$$(c-1)\{2f\varphi X-\eta(X)(\zeta-A\xi)+\langle\zeta-A\xi, X\rangle\xi\}=0$$

from which

$$2fX=3f(\langle\zeta, X\rangle\zeta+\eta(X)\xi)-f\langle A\xi, X\rangle\zeta+\eta(X)\varphi A\xi\tag{3.4}$$

for any tangent vector X of M, because of $c\neq 1$.

Substituting $W=Z=E_i$ into (3. 3) and summing up i from 1 to $2m$, we obtain

$$\begin{aligned}
&(\operatorname{tr} A)\langle A^2X, Y\rangle-(\operatorname{tr} A^2)\langle AX, Y\rangle\\
&=(\frac{c+3}{4}+\frac{c-1}{12}f^2)((\operatorname{tr} A)\langle X, Y\rangle-2m\langle AX, Y\rangle)-(c-1)\langle\varphi A\varphi X, Y\rangle\\
&\quad-\frac{c-1}{3}\beta\langle X, Y\rangle+\frac{c-1}{2}(\langle A\zeta, X\rangle\langle\zeta, Y\rangle+\langle A\zeta, Y\rangle-(1+f^2)\langle AX, Y\rangle)\\
&\quad+\frac{c-1}{4}(\operatorname{tr} A)(\langle\zeta, X\rangle\langle\zeta, Y\rangle+\eta(X)\eta(Y))+\frac{(c-1)(m+1)}{3}f\langle\varphi X, Y\rangle
\end{aligned}\tag{3.5}$$

$$+\frac{(c-1)(3m+4)}{6}\eta(X)\langle A\xi,\ Y\rangle+\frac{(c-1)(m+2)}{6}\eta(Y)\langle A\xi,\ X\rangle$$

$$-\frac{(c-1)(3m+1)}{6}\eta(X)\langle \zeta,\ Y\rangle-\frac{(c-1)(m-1)}{6}\eta(Y)\langle \zeta,\ X\rangle$$

for any tangent vectors $X,\ Y$ of M, where $\beta:=\langle A\xi,\ \xi\rangle$

Substituting $X=\xi$ into (3. 5), we see that

$$\begin{aligned}(3.6)\qquad &(\mathrm{tr}\ A)A^2\xi-(tr\ A^2)A\xi\\ &=(\frac{c+3}{4}+\frac{c-1}{12}f^2)((\mathrm{tr}\ A)\xi-2mA\xi-(c-1)f\varphi A\zeta+\frac{c-1}{6}\{3\gamma+(5m+3)f^2\\ &\quad-(3m+1)\}\zeta-\frac{c-1}{6}\{(3m+7)f^2-(3m+1)\}A\xi+\frac{c-1}{12}\{2m\beta-3(\mathrm{tr}\ A)(1-f^2)\}\xi,\end{aligned}$$

where $\gamma:=\langle A\zeta,\ \xi\rangle$. Substituting $Y=\xi$ into (3.5), we find

$$\begin{aligned}(3.7)\qquad &(\mathrm{tr}\ A)A^2\xi-(tr\ A^2)A\xi\\ &=(\frac{c+3}{4}+\frac{c-1}{12}f^2)((\mathrm{tr}\ A)\xi-2mA\xi)-(c-1)f\varphi A\zeta\\ &\quad+\frac{c-1}{6}\{3\gamma-(m+3)f^2-(m-1)\}\zeta\\ &\quad-\frac{c-1}{6}\{(m+5)f^2-(m-1)\}A\xi+\frac{c-1}{12}\{2(3m+2)\beta-3(\mathrm{tr}\ A)(1-f^2)\}\xi.\end{aligned}$$

From (3.6) and (3.7), we obtain

$$(3.8)\qquad (1-f^2)A\xi=(1-3f^2)\zeta+\beta\xi,$$

because of $c\neq 1$. From (3.4) and (3.8), it follows that

$$(3.9)\qquad f\{(1-f^2)X-\langle\zeta,\ X\rangle\zeta-\eta(X)\xi\}=0$$

for any tangent vector X of M.

Let M_0 be a set consisting of points of M at which the function $1-f^2$ does not vanish. By virtue of Lemma 1, M_0 is a nonempty open set in M. There exists a nonzero tangent vector X at each point of M_0 such that $\langle X,\ \zeta\rangle=\ \langle X,\ \xi\rangle=0$, because $\dim M\geq 4$. Thus, from (3.9), we can see that the function f vanishes identically on M_0. Since M_0 is open and closed, we find $M_0=M$. Consequently the structure vector field $\tilde{\xi}$ of $\tilde{M}(c)$ tangent to M. Q. E. D.

THEOREM 2. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$, of dimension $2m+1$ (≥ 5). Then the structure tensor φ induced on M and the second fundamental tensor A derived from the unit normal commute each other, that is, $\varphi A=A\varphi$.*

PROOF. Combining (2.14) with Theorem 1 and using Lemma 2, we obtain

$$(\varphi A-A\varphi)X=\nabla_X\zeta-A\varphi X=\nabla_X A\xi-A\nabla_X\xi=(\nabla_X A)\xi=0$$

for any vector X tangent to M. Q. E. D.

LEMMA 4. *Let M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$,* *of dimension* $2m+1$ (≥ 5). *Then it follows that*

$$A\zeta=\alpha\zeta+\xi, \tag{3.10}$$

(3.11) α *is constant on* M

and

$$A^2X=\alpha AX+\frac{c+3}{4}X-\frac{c-1}{4}\langle\zeta, X\rangle\zeta+\eta(X)\xi \tag{3.12}$$

for any tangent vector X *of* M, *where* $\alpha:=\langle A\zeta, \zeta\rangle$.

PROOF. By Theorem 1 and Theorem 2, (3.5) reduces to

$$\begin{aligned}&(\mathrm{tr}\ A)\langle A^2, X, Y\rangle-(\mathrm{tr}\ A^2)\ \langle AX, Y\rangle\\ &=\frac{c+3}{4}\{(\mathrm{tr}\ A)\langle X, Y\rangle-2m\langle AX, Y\rangle\}\\ &\quad+\frac{c-1}{2}\{\langle AX, Y\rangle-\langle A\zeta, X\rangle\langle\zeta, Y\rangle+\langle A\zeta, Y\rangle\langle\zeta, X\rangle-\langle\zeta, X\rangle\eta(Y)\\ &\quad+\langle\zeta, Y\rangle\eta(X)\}-\frac{c-1}{4}(\mathrm{tr}\ A)\{\langle\zeta, X\rangle\langle\zeta, Y\rangle+\eta(X)\eta(Y)\}\end{aligned} \tag{3.13}$$

for any tangent vectors X, Y of M. Interchanging the role of X and Y in (3. 13), we see that

$$\langle\zeta, X\rangle A\zeta+\eta(X)\zeta=\langle A\zeta, X\rangle\zeta+\langle\zeta, X\rangle\xi \tag{3.14}$$

for any tangent vector X of M. Substituting $X=\xi$ into this equation, we find (3.10). Combining (3.1) with (3.10) and using (2.14), it follows that

$$X\alpha=(\nabla_X h)(\zeta, \zeta)+2\langle A\zeta, \nabla_X\zeta\rangle=2\langle\alpha\zeta+\xi, \varphi AX\rangle=0 \tag{3.15}$$

for any tangent vector X of M. That is , α is a constant on M. Differentiating (3.10) covariantly with any tangent vector X of M, and using (2. 14) and (3. 1), we obtain (3.12).

Q. E. D.

PROPOSITION 3. *Let* M be a cyclic parallel hypersurface in a Sasakian space form $\tilde{M}(c)$, $c\neq 1$, *of dimension* $2m+1$ (≥ 5). *If* $\alpha^2+c+3=0$ *on* M, *we have*

$$AX=\frac{\alpha}{2}(X+\langle\zeta, X\rangle\zeta-\eta(X)\xi)+\langle\zeta, X\rangle\xi+\eta(X)\zeta \tag{3.16}$$

for any tangent vector X of M.

PROOF. In this case, by Lemma 4, we see that M has three constant princical curvatures $\frac{\alpha}{2}$, $\frac{\alpha+\sqrt{\alpha^2+4}}{2}$ and $\frac{\alpha-\sqrt{\alpha^2+4}}{2}$. Their multiplicities are $2m-2$, 1 and 1 respectively. Therefore we obtain (3. 16).

Q. E. D.

We have the following corollary of Proposition 3 :

COROLLARY 4. *Let* M *be a cyclic parallel hypersurface in a Sasakian space form* $\tilde{M}(-3)$ *of dimension* $2m+1$ (≥ 5). *If there exists a point* x *of* M *satisfying* $\alpha(x)=0$, *then* M *is totally contact geodesic.*

PROPOSITION 5. *Let* M *be a parallel hypersurface in a Sasakian space form* $\tilde{M}(1)$. *If the function* f *is a constant on* M, *then it follows that*

$$A\xi=\zeta, \tag{3.17}$$

(3.18) $A\zeta=\alpha\zeta+\xi$,

(3.19) $f=0$ (*i. e.*, $\tilde{\xi}$ *is tangent to* M),

(3.20) $\varphi A=A\varphi$

(3.21) α *is a constant on* M

and

(3.22) $A^2X=\alpha AX+X$

for any tangent vector X *of* M, *where* $\alpha := \langle A\zeta, \zeta\rangle$.

PROOF. Since f is a constant on M, (3.17) is obvious. Substituting $Y=\zeta$ and $W=Z=\xi$ into (3.3) and using (3.17), we obtain

$$(1-f^2)A\zeta=\alpha\zeta+(1-f^2)\xi.$$

By Lemma 1, $1-f^2$ is a positive constant on M. Thus

$$A\zeta=\frac{\alpha}{1-f^2}\zeta+\xi. \tag{3.23}$$

Further, since M is parallel, we get

$$\begin{aligned} 0 &= (\nabla_X A)\xi \\ &=\nabla_X A\xi-A\nabla_X\xi \\ &=\nabla_X\zeta-A(\varphi X+fAX) \\ &=(\varphi A-A\varphi)X-f(A^2X+X) \end{aligned} \tag{3.24}$$

for any tangent vector X of M. Substituting $X=\xi$ into (3.24), we have

$$f\left(\frac{\alpha}{1-f^2}\zeta+2\xi\right)=0. \tag{3.25}$$

This shows (3.19), from which (3.18) and (3.20) are obtained. By a similar argument as the proof of Lemma 4, we obtain (3.21) and (3.22). Q. E. D.

§4. Hypersurfaces with $\varphi A=A\varphi$ in a Sasakian space form.

Let M be a hypersurface with $\varphi A=A\varphi$ in a Sasakian manifold $\tilde{M}$. We can see that $M_0 := \{x\in M \mid f^2(x)\neq 1\}$ is a nonempty open set in M, $\zeta\neq 0$ and $\xi\neq 0$ everywhere on M_0. From simple calculations, we get

$$\begin{aligned} &A\zeta=\alpha\zeta+\gamma\xi, \\ &A\xi=\gamma\zeta+\beta\xi, \\ &f(\alpha-\beta)=0 \text{ and } f\gamma=0 \text{ on } M_0, \end{aligned} \tag{4.1}$$

where $\alpha:=\frac{\langle A\zeta, \zeta\rangle}{1-f^2}$, $\beta:=\frac{\langle A\xi, \xi\rangle}{1-f^2}$ and $\gamma:=\frac{\langle A\zeta, \xi\rangle}{1-f^2}=\frac{\langle A\xi, \zeta\rangle}{1-f^2}$.

LEMMA 5. *Let* M *be a hypersurface with* $\varphi A=A\varphi$ *in a Sasakian manifold* $\tilde{M}$. *If* f *is a constant function on* M, *then it follows that*

(4.2) $A\xi=\zeta$, $f=0$, $A\zeta=\alpha\zeta+\xi$,

(4.3) $X\alpha=(\zeta\alpha)\langle\zeta, X\rangle$,

$$A^2X=\alpha AX+\frac{c+3}{4}X-\frac{c-1}{4}(\langle\zeta, X\rangle\zeta+\eta(X)\xi), \tag{4.4}$$

$$(\nabla_X A)\xi=0 \tag{4.5}$$

and

$$(\nabla_X A)\zeta=-\frac{c-1}{4}\varphi X+(\zeta\alpha)\langle\zeta, X\rangle\zeta \tag{4.6}$$

for any tangent vector X of M.

PROOF. From (2.14) and (4.1), we have (4.2) everywhere on M. Differentiating $A\xi$ covariantly with any tangent vector X of M and using (2.14) and (4.2), we obtain

$$(\nabla_X A)\zeta=-\varphi A^2X+\alpha\varphi AX+\varphi X+(X\alpha)\zeta. \tag{4.7}$$

Using the equation of Codazzi (2.17) and (4.7), we have

$$(X\alpha)\zeta-(\nabla_\zeta A)X=\varphi A^2X-\alpha\varphi AX-\frac{c+3}{4}\varphi X, \tag{4.8}$$

from which

$$(X\alpha)\langle\zeta, Y\rangle-(Y\alpha)\langle\zeta, X\rangle=2\{\langle\varphi A^2X, Y\rangle-\alpha\langle\varphi AX, Y\rangle-\frac{c+3}{4}\langle\varphi X, Y\rangle\} \tag{4.9}$$

for any tangent vectors X, Y of M. Substituting $Y=\zeta$ into (4.9), we find (4.3). Substituting (4.3) into (4.9), we have

$$\varphi A^2X-\alpha\varphi AX-\frac{c+3}{4}\varphi X=0 \tag{4.10}$$

for any tangent vector X of M. From (2.14) and (4.10), we obtain (4.4). (4.5) is obvious. From (4.3), (4.4) and (4.7), we have (4.6). Q. E. D.

PROPOSITION 6. *Let M be a hypersurface with $\varphi A=A\varphi$ in a Sasakian space form $\tilde{M}(c)$, $c\neq-3$, of dimension $2m+1(\geq 5)$. If f is a constant function on M, then M is cyclic parallel.*

PROOF. Differentiating (4.4) covariantly with ζ, we find

$$(\zeta\alpha)(AX-\langle\alpha\zeta+\xi, X\rangle\zeta-\langle\zeta, X\rangle\xi)=0 \tag{4.11}$$

for any tangent vector X of M.

Let M_1 be a set consisting points of M at which the functoin $\zeta\alpha$ does not vanish, and suppose that M_1 is not empty. From (4. 11), it follows that

$$AX=\langle\alpha\zeta+\xi, X\rangle\zeta+\langle\zeta, X\rangle\xi$$

for any tangent vector X of M_1. Combining this with (4. 4), we find

$$(c+3)(X-\langle\zeta, X\rangle\zeta-\eta(X)\xi)=0$$

for any tangent vector X of M_1. Thus the assumption of M_1 produces a contradiction because $c\neq-3$ and dim $M\geq 4$. Accordingly we obtain

$$\zeta\alpha=0 \text{ (everywhere on } M). \tag{4.12}$$

Therefore, from this and (4.3) we see that α is a constant on M. Using this fact, (4.6) reduces to

$$(\nabla_X A)\zeta=-\frac{c-1}{4}\varphi X \tag{4.13}$$

for any tangent vector X of M.

Differentiating (4.4) covariantly with any tangent vector Y of M, we obtain

$$(\nabla_Y A)AX+A(\nabla_Y A)X=\alpha(\nabla_Y A)X-\frac{c-1}{4}\{\langle\varphi AY,\ X\rangle+\langle\zeta,\ X\rangle\varphi AY+\langle\varphi Y,\ X\rangle\xi+\eta(X)\varphi Y\}. \tag{4.14}$$

Interchanging the role of X and Y in the above equation and combing these equations with the equation of Codazzi (2.17), we get

$$\begin{aligned}&(\nabla_X A)AY-(\nabla_Y A)AX\\&=\frac{c-1}{4}\{\langle\alpha\zeta+\xi,\ X\rangle\varphi Y-\langle\alpha\zeta+\xi,\ Y\rangle\varphi X-2\langle\varphi AX,\ Y\rangle\zeta\},\end{aligned} \tag{4.15}$$

from which

$$\begin{aligned}&(\nabla_X A)AY-A(\nabla_X A)Y\\&=\frac{c-1}{4}\{\langle\varphi X,\ Y\rangle(\alpha\zeta+\xi)-\langle\alpha\zeta+\xi,\ Y\rangle\varphi X-\langle\varphi AX,\ Y\rangle\zeta+\langle\zeta,\ Y\rangle\varphi AX\}.\end{aligned} \tag{4.16}$$

From (4.14) and (4.16), we have

$$2(\nabla_X A)AY=\alpha(\nabla_X A)Y+\frac{c-1}{4}\{\alpha\langle\varphi X,\ Y\rangle\zeta-\alpha\langle\zeta,\ Y\rangle\varphi X-2\langle\varphi AX,\ Y\rangle\zeta-2\eta(Y)\varphi X\} \tag{4.17}$$

for any tangent vectors X, Y of M. Combining this with (4.4), it follows that

$$(\alpha^2+c+3)\{(\nabla_X A)Y+\frac{c-1}{4}\{\langle\varphi X,\ Y\rangle\zeta+\langle\zeta,\ Y\rangle\varphi X)\}=0 \tag{4.18}$$

for any tangent vectors X, Y of M. Thus M is cyclic parallel provided that $\alpha^2+c+3\neq 0$.

Next, assuming that $\alpha^2+c+3=0$, (4. 2) and (4. 4) show that M has three constant principal curvatures $\frac{\alpha}{2}$, $\frac{\alpha+\sqrt{\alpha^2+4}}{2}$, and $\frac{\alpha-\sqrt{\alpha^2+4}}{2}$. Their multiplicities are $2m-2$, 1, and 1 respectively. This gives

$$AX=\frac{\alpha}{2}(X+\langle\zeta,\ X\rangle\zeta-\eta(X)\xi)+\langle\zeta,\ X\rangle\xi+\eta(X)\zeta \tag{4.19}$$

for any tangent vector X of M. Differentiating this covariantly, we find

$$\begin{aligned}(\nabla_X A)Y&=(\frac{\alpha^2}{4}+1)(\langle\varphi X,\ Y\rangle\zeta+\langle\zeta,\ Y\rangle\varphi X)\\&=-\frac{c-1}{4}(\langle\varphi X,\ Y\rangle\zeta+\langle\zeta,\ Y\rangle\varphi X)\end{aligned} \tag{4.20}$$

for any tangent vectors X, Y of M. Thus M is cyclic parallel because of Lemma 2.

Q. E. D.

From the proof of Proposition 6, we have the following

REMARK. *Let M be a hypersurface with $\varphi A=A\varphi$ in a Ssasakian space form $\tilde{M}(-3)$. If f is a constant and $\zeta\alpha=0$ everywhere on M, then M is cyclic parallel.*

Let M be a hypersurface with $\varphi A = A\varphi$ in a Ssasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$ and M_2 a set consisting of points of M at which $0<f^2<1$. Assume that f is a nonconstant function on M, then M_2 is a nonempty open set in M because of Lemma 1. Thus M_2 is a hypersurface (not necessarily connected) in $\tilde{M}(c)$, and we have

THEOREM 7. *Let M be a hypersurface with $\varphi A = A\varphi$ in a Ssasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$. If f is a nonconstant function on M, then M_2 is a totally umbilical hypersurface. Therefore $c=1$ and M_2 is parallel* (*and the mean curvature on each connected component of M_2 is a constant*).

PROOF. For this case,

(4.21) $\quad A\zeta = \alpha\zeta$

and

(4.22) $\quad A\xi = \alpha\xi$

on M_2, where $\alpha := \frac{\langle A\zeta, \zeta\rangle}{1-f^2} = \frac{\langle A\xi, \xi\rangle}{1-f^2}$. Differentiating (4. 21) and (4. 22) covariantly with any tangent vector X of M_2, and using (2. 14), we obtain

(4.23) $\quad (\nabla_X A)\zeta = (X\alpha)\zeta + \alpha(\varphi AX - f\,X) - \varphi A^2X + fAX$

and

(4.24) $\quad (\nabla_X A)\xi = (X\alpha)\xi + \alpha(\varphi X + fAX) - \varphi AX - fA^2X.$

Combining (5.24) with the equation of Codazzi (2.17), it follows that

$$\begin{aligned}(\nabla_\xi A)X = {} & (X\alpha)\xi + \alpha(\varphi X + fAX) - \varphi AX - fA^2X \\ & - \frac{c-1}{4}f\{(1-f^2)X + 3\langle\zeta, X\rangle\zeta - \eta(X)\xi\}\end{aligned} \tag{4.25}$$

for any tangent vector X of M_2. Taking the inner product with ξ, we have

(4.26) $\quad (1-f^2)(X\alpha) = (\xi\alpha)\eta(X)$

for any tangent vector X of M_2. Substituting $X=\zeta$ into (4. 25), we find

(4.27) $\quad \zeta\alpha = 0$ and $\xi\alpha = -(c-1)f(1-f^2)$ on M_2.

From (4.26) and(4.27), we get

(4.28) $\quad X\alpha = -(c-1)f\,\eta(X)$

for any tangent vector X of M_2. Using this equation, (4.25) reduces to

(4.29) $\quad (\nabla_\xi A)X = \alpha(\varphi X + fAX) - \varphi AX - fA^2X - \frac{c-1}{4}f\{(1-f^2)X + 3\langle\zeta, X\rangle\zeta + 3\eta(X)\xi\}$

for any vector X tangent to M_2. Taking the inner product with any tangent vector Y of M_2 and interchanging the role of X and Y, we find

(4.30) $\quad AX = \alpha X$

for any tangent vector X of M_2. Thus M_2 is a totally umbilical hypersurface with mean curvature $\rho = \alpha$. Therefore $c=1$, M_2 is a parallel hypersurface and the mean curvature a constant on each connected component of M_2, by virture of Lemma 3. Q. E. D.

PROPOSITION 8. *Let M be a hypersurface with $\varphi A = A\varphi$ in a Ssasakian space form $\tilde{M}(c)$*

of dimension $2m+1(\geq 5)$. *If* f *is a nonconstant function on* M, *then* M *is parallel and* $c=1$.

PROOF. By Lemma 2, $M_2 := \{x \in M \mid 0 < f^2(x) < 1\}$ is a parallel hypersurface in $\tilde{M}(c)$ and $c=1$. We put $M_3 := \{x \in M \mid \| \nabla A \| (x) \neq 0\}$. Suppose that M_3 is not empty, then since $M_3 \subset M - M_2$, it follows that

$$f(x)=0 \quad \text{or} \quad f^2(x)=1 \quad \text{for any } x \in M_3.$$

If there exists some point x of M_3 satisfying $f^2(x)=1$, we see that $U \cap M_2 \neq \phi$ for any neighborhood U of x in M, i. e., x is an accumulation point of M_2, because of Lemma 1. Thus we have $\| \nabla A \| (x)=0$. This is a contradiction.
Therefore we obtain $M_3 \subset \{x \in M \mid f(x)=0\}$. In this case, by Proposition 6, M_3 is cyclic parallel. Since $c=1$, M_3 is parallel. Therefore the assumption of M_3 produces a contradiction. Accordingly M is parallel. Q. E. D.

Proposition 6 and Proposition 8 assert the following

THEOREM 9. *Let* M *be a hypersurface with* $\varphi A = A\varphi$ *in a Sasakian space form* $\tilde{M}(c)$, $c \neq -3$, *of dimension* $2m+1$ (≥ 5). *Then* M *is a cyclic parallel.*

THEOREM 10. *Let* M *be a complete hypersurface with* $\varphi A = A\varphi$ *in a Sasakian space form* $\tilde{M}(c)$ *of dimension* $2m+1(\geq 5)$. *If* f *does not vanish everywhere on* M, *then* M *is a totally umbilical hypersurface with constant mean curvature, isometric to an ordinary sphere, and* $c=1$.

PROOF. By Lemma 5, f is a nonconstant function on M. Using Lemma 1, we see that any point of M is an accumulation point of M_2. Thus M is a totally umbilical hypersurface with constant mean curvature $\rho=\alpha$, by virtue of Theorem 7. In this case, we have

$$(\nabla\nabla f)(X, Y) := (\nabla_X df)Y = -(1+\alpha^2)f\langle X, Y\rangle$$

for any tangent vector fields X, Y on M. By virtue of Obata's theorem [6], we see that M is isometric to an ordinary sphere of radius $\sqrt{1+\alpha^2}$. Q. E. D.

§5. Cyclic parallel and totally contact umbilical hypersurfaces.

Let M be a totally contact umbilical hypersurface in a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$ and ρ the mean curvature of M in $\tilde{M}$. Then the second fundamental form h has the following form :

$$(5.1) \qquad \begin{aligned} &h(\xi, X) = \langle \zeta, X\rangle, \\ &h(X, Y) = \alpha\{\langle X, Y\rangle - \eta(X)\eta(Y)\} + \eta(X)h(\xi, Y) + \eta(Y)h(\xi, X) \\ &\qquad\quad = \alpha\{\langle X, Y\rangle - \eta(X)\eta(Y)\} + \eta(X)\langle \zeta, Y\rangle + \eta(Y)\langle \zeta, X\rangle, \end{aligned}$$

where $\alpha := \frac{2m}{2m-1}\rho$. (5. 1) is equivalent to

$$(5.2) \qquad A\xi = \zeta, \quad A\zeta = \alpha\zeta + \xi \quad \text{and} \quad \varphi A = \alpha\varphi \; (=A\varphi).$$

PROPOSITION 11. *Let M be a totally contact umbilical hypersurface in a Sasakian manifold $\tilde{M}$. Then M is cyclic parallel if and only if the mean curvature ρ of M in $\tilde{M}$ is a constant. In this casa, we have*

$$(\nabla_X h)(Y, Z) = \langle\varphi X, Y\rangle\langle\xi, Z\rangle + \langle\varphi X, Z\rangle\langle\xi, Y\rangle.$$

PROOF. Differentiating (5. 1) covariantly and making use of (2.14), we find

$$\begin{aligned}(\nabla_X h)(Y, Z) = &-\alpha\{\langle\varphi X, Y\rangle\eta(Z) + \langle\varphi X, Z\rangle\eta(Y)\} + \langle\varphi X, Y\rangle\langle\xi, Z\rangle + \langle\varphi X, Z\rangle\langle\xi, Y\rangle \\ &+\langle\varphi AX, Y\rangle\eta(Z) + \langle\varphi AX, Z\rangle\eta(Y) + (X\alpha)(\langle Y, Z\rangle - \eta(Y)\eta(Z)). \\ =&\langle\varphi X, Z\rangle\langle\xi, Z\rangle + \langle\varphi X, Z\rangle\langle\xi, Y\rangle + (X\alpha)(\langle Y, Z\rangle - \eta(Y)\eta(Z)).\end{aligned} \tag{5.3}$$

From this, we get

$$\begin{aligned}&(\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X) + (\nabla_Z h)(X, Y) \\ &= (X\alpha)(\langle Y, Z\rangle - \eta(Y)\eta(Z)) + (Y\alpha)(\langle Z, X\rangle - \eta(Z)\eta(X)) \\ &\quad + (Z\alpha)(\langle X, Y\rangle - \eta(X)\eta(Y)).\end{aligned} \tag{5.4}$$

If M is cyclic parallel, we obtain

$$X\alpha = \xi\alpha = 0 \quad (X \perp \xi). \tag{5.5}$$

Thus we see that α is a constant, i. e., ρ is a constant.

Conversely, assume that ρ is a constant. From (5. 4), we see that

$$(\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X) + (\nabla_Z h)(X, Y) = 0,$$

that is, M is cyclic parallel. Q. E. D.

PROSITION 12. *Let M be a totally contact umbilical hypersurface in a Sasakian space form $\tilde{M}(c)$ of dimension $2m+1(\geq 5)$. Then $c=-3$ and M is cyclic parpllel.*

PROOF. Since M is totally contact umbilical, we have

$$(\nabla_X A)Y = (X\alpha)(Y - \eta(Y)\xi) + \langle\varphi X, Y\rangle\xi + \langle\xi, Y\rangle\varphi X \tag{5.6}$$

for any tangent vectors X, Y of M, where $\alpha := \frac{2m}{2m-1}\rho$. Using the equation of Codazzi (2.17), we obtain

$$\begin{aligned}&(X\alpha)(Y - \eta(Y)\xi) - (Y\alpha)(X - \eta)(X)\xi) \\ &= \frac{c+3}{4}(\langle\xi, X\rangle\varphi Y - \langle\xi, Y\rangle\varphi X + 2\langle X, \varphi Y\rangle\xi)\end{aligned} \tag{5.7}$$

for any tangent vectors X, Y of M. Since dim $M \geq 4$, there exists a nonzero tangent vector X such that $\langle\xi, X\rangle = \eta(X) = 0$. Therefore, substituting $Y = \xi$ into (5.7) and taking the inner product with φX, we have $c=-3$. From this and (5.7), we have

$$(X\rho)(\langle Y, Z\rangle - \eta(Y)\eta(Z)) = (Y\rho)(\langle X, Z\rangle - \eta(X)\eta(Z)) \tag{5.8}$$

for any vectors X, Y, Z tangent to M. Sustituting $Y = Z = \xi$ into (5.8), we see that ρ is a constant. By virtue of Proposition 11, M is cyclic parallel. Q. E. D.

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