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Infinitesimal Affine Transformations in Finsler Geometry

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Abstract

In general, a vector field X on a differentiable manifold M with a linear connection ∇ is called an infinitesimal affine transformation if

(0.1) $$L_X\cdot\nabla_Y-\nabla_Y\cdot L_X-\nabla_{[X,\ Y]}=0,$$

for every vector field Y on M. A Finsler connection is regarded as a linear connection and Finsler vector fields can be regarded as fields on the tangent bundle. This paper will apply the above definition of infinitesimal affine transformations to Finsler geometry.

§ 1. Finsler vector fields and Finsler connections.

Let M be a differentiable manifold and $L(x, y)$ be a Finslerian fundamental function, a real value function of the bundle of all the non-zero tangent vectors which is denoted simply by TM and called the tangent bundle of M. As is well-known, a non-linear connection is a distribution on TM such that

(1.1) $$T_y(TM)=N_y+V_y \text{ (direct sum)},$$

where V_y is a vertical subspace of the tangent space $T_y(TM)$ at $y\in TM$. With respect to the induced coordinates x^i, y^i; $\partial/\partial y^i$ are a basis of V_y and $\delta/\delta x^i=\partial/\partial x^i-N^i_j\,\partial/\partial y^j$ form a basis of N_y. Then we can define an isomorphism I_y: $T_y\rightarrow T_y$ by

(1.2) $$I_y\colon \delta/\delta x^i \to \partial/\partial y^i \text{ and } \partial/\partial y^i \to \delta/\delta x^i.$$

As will be utilized later, also

(1.3) $$[\delta_j, \delta_k] = (\delta N^i_j/\delta x^k - \delta N^i_k/\delta x^j)\,\dot{\partial}_i,$$

$$[\delta_j, \dot{\partial}_k] = (\partial N^i_j/\partial y^k)\,\dot{\partial}_i, \quad [\dot{\partial}_j, \dot{\partial}_k] = 0.$$

Definition 1.

A Finsler vector field is a vector field on TM which is horizontal at each point. Therefore a Finsler vector field X is written as $X = X^i(x,y)\,\delta_i$ and $IX = X^i(x,y)\,\dot{\partial}_i$ is a vertical vector field on TM.

Let $\mathfrak{F}$ be the algebra of all the differentiable functions on TM, $\mathfrak{X}$ the $\mathfrak{F}$-module of all the vector fields on TM, and $\mathfrak{X}^h$ ($\mathfrak{X}^v$) be the horizontal (resp. vertical) submodule of $\mathfrak{X}$.

Definition 2.

A Finsler connection of M is a linear connection $\tilde{\nabla}$ of TM such that

(1.4) $$\tilde{\nabla}_X \mathfrak{X}^h \subset \mathfrak{X}^h,\ \tilde{\nabla}_X \mathfrak{X}^v \subset \mathfrak{X}^v \text{ and } \tilde{\nabla}_X(IY) = I(\tilde{\nabla}_X Y),$$

for any $X,\ Y \in \mathfrak{X}$.

Such a connection is determined by the set of functions $(\Gamma^i_{jk},\ C^i_{jk})$ defined by

(1.5) $$\tilde{\nabla}_{\delta_j}\delta_k = \Gamma^i_{jk}\delta_i \text{ and } \tilde{\nabla}_{\dot{\partial}_j}\dot{\partial}_k = C^i_{jk}\,\dot{\partial}_i.$$

We could regard a Finsler connection as two covariant differentiations $(\nabla, \dot{\nabla})$ on $\mathfrak{X}^h$ defined by

(1.6) $$\nabla_X Y = \tilde{\nabla}_X Y \text{ and } \dot{\nabla}_X Y = \tilde{\nabla}_{IX} Y \text{ for } X,\ Y \in \mathfrak{X}^h.$$

The torsion and the curvature of the linear connection $\tilde{\nabla}$ are given as

(1.7) $$\tilde{T}(X,\ Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,\ Y],$$

(1.8) $$\tilde{R}(X, Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,\ Y]}Z,$$

for $X,\ Y,\ Z \in \mathfrak{X}$. Then five Finsler tensors are obtained from (1.7)

(1.9) $$\tilde{T}(\delta_k, \delta_j) = T^i_{jk}\delta_i + R^i_{jk}\,\dot{\partial}_i,$$

(1.10) $$\tilde{T}(\dot{\partial}_k, \delta_j) = C^i_{jk}\delta_i + P^i_{jk}\,\dot{\partial}_i,$$

(1.11) $$\tilde{T}(\dot{\partial}_k, \dot{\partial}_j) = S^i_{jk}\,\dot{\partial}_i,$$

and three kinds of Finsler tensors from (1.8):

(1.12) $\tilde{R}(\delta_l, \delta_k)\delta_j = R^i_{jkl}\delta_i,$

(1.13) $\tilde{R}(\dot{\partial}_l, \delta_k)\delta_j = P^i_{jkl}\delta_i,$

(1.14) $\tilde{R}(\dot{\partial}_l, \dot{\partial}_k)\delta_j = S^i_{jkl}\delta_i,$

For simplicity we assume

(1.15) $T^i_{jk} = 0, \ S^i_{jk} = 0.$

§ 2. Infinitesimal affine transformations.

In the following we use only the canonical non-linear connection which appears in the Cartan and Berwald connections, and (1.15) is assumed.

We will look for the conditions of a Finsler vector field $X = X^i(x, y)\delta_i$ to be an infinitesimal affine transformation of a Finsler connection, that is, a linear connection $\tilde{\nabla}$ on the tangent bundle TM. It is well known that the conditions are written as

(2.1) $(L_X\tilde{\nabla})(Y,Z) = L_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y L_X Z - \tilde{\nabla}_{[X,\ Y]}Z = 0,$

for any $Y, Z \in \mathfrak{X}(TM)$. Because $L_X\tilde{\nabla}$ is a tensor field of order (1, 2), we have only to substitute the basis $(\delta_j, \dot{\partial}_k)$ to Y and Z.

First of all, we put $Y = \delta_j$ and $Z = \delta_k$. Then we have

(2.2) $[X^i\delta_i, \Gamma^l_{jk}\delta_l] - \nabla_{\delta_j}[X^i\delta_i, \delta_k] - \tilde{\nabla}_{[X^i\delta_i,\ \delta_j]}\ \delta_k = 0.$

Taking h- and v-components of (2.2), from (1.3) and (1.5) we get

(2.3) $\nabla_j\nabla_k X^i + X^l R^i_{kjl} = 0,$

(2.4) $\nabla_j(X^l R^i_{lk}) = 0.$

Substituting δ_i or $\dot{\partial}_i$ into Y and Z similarily, and taking h- and v-components, we have the following:

Theorem 1.

(I) $X = X^i\delta_i$ *is an infinitesimal affine transformation of a linear connection* $\tilde{\nabla}$ *on* TM *if and only if*

(**I***a*) $Y = \delta_j,\ Z = \delta_k,$

$$\nabla_j\nabla_k X^i + X^l R^i_{kjl} = 0, \quad \nabla_j(X^l R^i_{lk}) = 0.$$

(**I***b*) $Y = \delta_j,\ Z = \dot{\partial}_k,$

$$\nabla_j(\dot{\partial}_k X^i) = 0, \quad X^l R^i_{kjl} - \nabla_j(X^l P^i_{lk}) = 0.$$

(Ic) $Y = \dot{\partial}_j$, $Z = \delta_k$,

$$\dot{\nabla}_j \nabla_k X^i - X^l P^i_{klj} = 0, \quad \dot{\nabla}_j(X^l R^i_{lk}) = 0.$$

(Id) $Y = \dot{\partial}_j$, $Z = \dot{\partial}_k$,

$$\dot{\nabla}_j(\dot{\partial}_k X^i) = 0, \quad X^l R^i_{klj} + \dot{\nabla}_j(X^l P^i_{kl}) = 0.$$

(II) *A vertical vector field* $X = X^i \dot{\partial}_i$ *is an infinitesimal affine transformation of* $\tilde{\nabla}$ *if and only if*

(IIa) $Y = \delta_j$, $Z = \delta_k$,

$$X^l P^i_{kjl} + \nabla_j(X^l C^i_{kl}) = 0, \quad \nabla_j(\nabla_k X^i + X^l P^i_{kl}) = 0.$$

(IIb) $Y = \delta_j$, $Z = \dot{\partial}_k$,

$$\nabla_j \dot{\nabla}_k X^i + X^l P^i_{kjl} = 0.$$

(IIc) $Y = \dot{\partial}_j$, $Z = \delta_k$,

$$\dot{\nabla}_j(X^l C^i_{kl}) + X^l S^i_{kjl} = 0, \quad \dot{\nabla}_j(\nabla_k X^i + X^l P^i_{jlk}) = 0.$$

(IId) $Y = \dot{\partial}_j$, $Z = \dot{\partial}_k$,

$$\dot{\nabla}_j \dot{\nabla}_k X^i + X^l S^i_{kjl} = 0.$$

Now we are concerned with the Cartan connection. Contraction of y^k with the second equation of (Ib) leads us to

Lemma 1.

If a Finsler vector field $X = X^i \delta_j$ *is an infinitesimal affine transformation of Cartan's* $\tilde{\nabla}$, *then it satisfies*

(2.5) $$X^l R^i_{jl} = 0.$$

In general, the set of all infinitesimal affine transformations on a manifold with a linear connection is a Lie subalgebra. That is, if X and Y are affine, so is $[X, Y]$. In Finsler geometry we have

Theorem 2.

If Finsler vector fields X *and* Y *are infinitesimal affine transformations of Cartan's linear connection* ∇, *then a Finsler vector field* $h[X, Y]$ *is also affine.*

Proof. It is well known that $L_X \tilde{\nabla} = 0$ and $L_Y \tilde{\nabla} = 0$ yield

$$L_{[X, Y]} \tilde{\nabla} = 0.$$

Because of the formula $L_{X+Y} = L_X + L_Y$ we have

$$L_{h[X,\ Y]}\tilde{\nabla} + L_{v[X,\ Y]}\tilde{\nabla} = 0.$$

On the other hand, $v[X,\ Y] = X^i Y^j R^k_{ij}\dot{\partial}_k = 0$ by Lemma 1. Hence we have

$$L_{h[X,\ Y]}\tilde{\nabla} = 0.$$

If vector fields X and Y are vertical, then $[X,\ Y]$ is also vertical, that is, $h[X,\ Y]=0$. Similarily we have

Theorem 3.

If vertical vector fields X and Y are infinitesimal affine transformations of Cartan's linear connection $\tilde{\nabla}$, vertical vector field $[X,\ Y]$ is also affine.

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