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メタデータ	言語: English
	出版者: 北海道教育大学
	公開日: 2012-11-07
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URL	https://doi.org/10.32150/00004255

北海道教育大学紀要（第２部Ａ）第44巻　第２号
Journal of Hokkaido University of Education (Section II A) Vol.44, No.2

平成６年３月
March, 1994

Some remarks on the normalized metric tensor and the Lie derivatives in an areal space of the submetric class

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Abstract

In a previous paper [1], we investigated when the normalized metric tensor g_{ij} is homogeneous of degree zero with respect to p^i_α.

However if the original metric tensor $\overset{\circ}{g}_{ij}$ is assumed to be homogeneous of degree zero with respect to p^i_α,it follows that the normalized metric tensor g_{ij} is homogeneous of degree zero with respect to p^i_α. In this paper, we prove the above and exhibit some results on the Lie derivatives.

We employ the notations used in papers [1], [2] and [3] without reexplaining them in this paper.

§1. Remarks on the normalized metric tensor

Let $F(x^i, p^i_\alpha)$ be the fundamental function and $\overset{\circ}{g}_{ij}$ the original metric tensor in an areal space of the submetric class.

The normalized metric tensor g_{ij} is defined by

(1.1) $$g_{ij}=\left(\frac{1}{m}L^{\alpha\beta}_{ij}+p^\alpha_i p^\beta_j\right)b_{\alpha\beta},$$

$$b_{\alpha\beta}=\rho \overset{\circ}{g}_{ij}p^i_\alpha p^j_\beta,$$

where $\quad L^{\alpha\beta}_{ij}=F^{-1}F\,;\,{}^\alpha_i\,;\,{}^\beta_j-p^\alpha_i p^\beta_j+p^\alpha_j p^\beta_i, \qquad p^\alpha_i=F^{-1}F\,;\,{}^\alpha_i$

and ρ is determined so that $F^2=\det\,(b_{\alpha\beta})$ is satisfied, provided that $\det\,(b_{\alpha\beta})$ does not vanish identically.

Differentiating g_{ij} with respect to p^α_γ and contracting with p^α_δ, by virtue of the relations

$$p^\alpha_a p^a_\delta=\delta^\alpha_\delta, \qquad p^\alpha_i\,;\,{}^\gamma_a=L^{\alpha\gamma}_{ia}-p^\alpha_a p^\gamma_i, \qquad L^{\alpha\beta}_{ij}\,;\,{}^\gamma_a p^a_\delta=-L^{\gamma\beta}_{ij}\delta^\alpha_\delta-L^{\alpha\gamma}_{ij}\delta^\beta_\delta,$$

we have

$$g_{ij}\,;\,{}^\gamma_a p^a_\delta=\Big(\frac{1}{m}L^{\alpha\beta}_{ij}+p^\alpha_i p^\beta_j\Big)\Big(\rho \overset{\circ}{g}_{cd}\Big)\,;\,{}^\gamma_a p^c_\alpha p^d_\beta p^a_\delta, \tag{1.2}$$

from which, by virtue of the relations

$$L^{\alpha\beta}_{ij}p^i_\varepsilon=0, \qquad g_{ij}\,;\,{}^\gamma_a p^i_\varepsilon p^j_\rho b^{\varepsilon\rho}=0,$$

it follows that

$$b^{\alpha\beta}(\rho\overset{\circ}{g}_{cd})\,;\,{}^\gamma_a p^c_\alpha p^d_\beta p^a_\delta=0. \tag{1.3}$$

Consequently, if the original metric tensor $\overset{\circ}{g}_{ij}$ is assumed to be homogeneous of degree zero with respect to p^i_α, that is, $\overset{\circ}{g}_{cd}\,;\,{}^\gamma_a p^a_\delta=0$, it follows from (1.3) that

$$\rho\,;\,{}^\gamma_a p^a_\delta=0. \tag{1.4}$$

Hence from (1.2) and (1.4) we have

$$g_{ij}\,;\,{}^\gamma_a p^a_\delta=0,$$

provided that the original metric tensor $\overset{\circ}{g}_{ij}$ is homogeneous of degree zero with respect to p^i_α.

From (1.1) we have

$$C_{kh\mu}^{\lambda}=g_{ij}\,;\,{}^\lambda_h\gamma^i_k p^j_\mu=-\frac{1}{m}L^{\alpha\beta}_{kh}\delta^\lambda_\mu b_{\alpha\beta}+L^{\alpha\lambda}_{kh}b_{\alpha\mu}, \tag{1.5}$$

where $\quad \gamma^i_k=\delta^i_k-\beta^i_k, \qquad \beta^i_k=p^i_\varepsilon p^\varepsilon_k.$

Putting $\lambda=\mu$ in (1.5), we have

$$C_{kh\lambda}^{\lambda}=0. \tag{1.6}$$

On the other hand, by virtue of $\gamma^i_j p^j_\mu=0$, we have

$$C_{kh\mu}^{\lambda}=b_{\alpha\mu}L^{\alpha\lambda}_{kh}-g_{ih}\gamma^i_k\delta^\lambda_\mu$$

from which, contracting with $b^{\mu\nu}$, we have

$$L^{\alpha\beta}_{ij}=g''_{ij}b^{\alpha\beta}+C_{ij\gamma}^{\beta}b^{\alpha\gamma}, \tag{1.7}$$

where $\quad g''_{ij}=\gamma^h_i g_{hj}.$

From (1.1) and (1.7), we have respectively

$$g_{ij|h}=\frac{1}{m}L^{\alpha\beta}_{ij|h}b_{\alpha\beta}, \tag{1.8}$$

$$L^{\alpha\beta}_{ij|h}=-\beta^a_b\Gamma^b_{ah}\,;\,{}^\alpha_i\,;\,{}^\beta_j \tag{1.9}$$

from which, we have

Theorem. $g_{ij|h}=0$ *is equivalent to* $\beta^a_b\Gamma^b_{ah}\,;{}^\alpha_i\,;{}^\beta_j\,b_{\alpha\beta}=0$.

§2. Lie derivatives with respect to a concurrent vector

Consider an infinitesimal extended point transformation [1]

(2.1) $$\bar{x}^i=x^i+\xi^i d\tau, \qquad \bar{p}^i_\alpha=p^i_\alpha+\xi^i{}_{,j}p^j_\alpha d\tau.$$

From the definition of the Lie derivatives with respect to ξ^i, we have

(2.2) $$\underset{\xi}{\mathcal{L}}p^i_\alpha=0,$$

(2.3) $$\underset{\xi}{\mathcal{L}}g_{ij}=g_{aj}\xi^a{}_{|i}+g_{ia}\xi^a{}_{|j}+g_{ij}\,;{}^\alpha_a\,\xi^a{}_{|b}p^b_\alpha,$$

(2.4) $$\underset{\xi}{\mathcal{L}}p^\alpha_i=\beta^a_b\,;{}^\alpha_i\,\xi^b{}_{|a}=\gamma^a_i p^b_\beta b^{\alpha\beta}\underset{\xi}{\mathcal{L}}g_{ab},$$

(2.5) $$\underset{\xi}{\mathcal{L}}F=F\beta^a_b\xi^b{}_{|a}=\frac{1}{2}Fb^{\alpha\beta}p^a_\alpha p^b_\beta\underset{\xi}{\mathcal{L}}g_{ab},$$

(2.6) $$\underset{\xi}{\mathcal{L}}\Gamma^i_{jk}=R^i_{jka}\xi^a+\Gamma^i_{jk}\,;{}^\alpha_a\,\xi^a{}_{|b}p^b_\alpha+\xi^i{}_{|j|k},$$

(2.7) $$\underset{\xi}{\mathcal{L}}C^i_{j,}{}^\alpha_a=\frac{1}{2}g^{ib}[\{g_{mn}\,;{}^\alpha_a\,\beta^c_e g^{en}(\delta^m_j\gamma^d_b-\delta^m_b\gamma^d_j)$$
$$-2\delta^c_b C^d_{j,}{}^\alpha_a\}\underset{\xi}{\mathcal{L}}g_{cd}+(\delta^c_b\gamma^d_j+\delta^c_j\beta^d_b)(\underset{\xi}{\mathcal{L}}g_{cd})\,;{}^\alpha_a].$$

For a general tensor T^h_{ij}, we have

(2.8) $$(\underset{\xi}{\mathcal{L}}T^h_{ij})_{|k}-\underset{\xi}{\mathcal{L}}(T^h_{ij|k})=-T^a_{ij}\underset{\xi}{\mathcal{L}}\Gamma^h_{ak}+T^h_{aj}\underset{\xi}{\mathcal{L}}\Gamma^a_{ik}+T^h_{ia}\underset{\xi}{\mathcal{L}}\Gamma^a_{jk}+T^h_{ij}\,;{}^\alpha_a\underset{\xi}{\mathcal{L}}\Gamma^a_{\alpha k},$$

(2.9) $$(\underset{\xi}{\mathcal{L}}T^h_{ij})|^\alpha_a-\underset{\xi}{\mathcal{L}}T^h_{ij}|^\alpha_a=-T^b_{ij}\underset{\xi}{\mathcal{L}}C^h_{b,}{}^\alpha_a+T^h_{bj}\underset{\xi}{\mathcal{L}}C^b_{i,}{}^\alpha_a+T^h_{ib}\underset{\xi}{\mathcal{L}}C^b_{j,}{}^\alpha_a,$$

(2.10) $$(\underset{\xi}{\mathcal{L}}T^h_{ij})\,;{}^\alpha_a=\underset{\xi}{\mathcal{L}}T^h_{ij}\,;{}^\alpha_a.$$

If the vector ξ^i in the infinitesimal extended point transformation (2.1) is assumed to be a concurrent vector [2], by definition it satisfies the following relations

(2.11) $$\xi^i{}_{|j}=-\delta^i_j, \qquad \xi^i|^\alpha_a=0, \qquad \xi^b C^i_{b,}{}^\alpha_a=0$$

from which, it follows that using the results (2.3) — (2.7)

(2.12) $$\underset{\xi}{\mathcal{L}}g_{ij}=-2g_{ij},$$

(2.13) $$\underset{\xi}{\mathcal{L}}p^\alpha_a=0,$$

(2.14) $$\underset{\xi}{\mathcal{L}}F=-mF,$$

(2.15) $$\underset{\xi}{\mathcal{L}}\Gamma^i_{jk}=R^i_{jka}\xi^a,$$

(2.16) $$\underset{\xi}{\mathcal{L}}C^i_{j,}{}^\alpha_a=0.$$

From (2.16) and (2.9), we have

$$(\underset{\xi}{\pounds} T^h_{ij})|^{\alpha}_a = \underset{\xi}{\pounds} T^h_{ij}|^{\alpha}_a.$$

Hence we have

Theorem. *If the vector ξ^i in the infinitesimal extended point transformation* (2.1) *is a concurrent vector, the operations of the Lie derivative with respect to ξ^i and the covariant derivative with respect to p^i_α* [3] *are commutative.*

If the vector ξ^i in (2.1) is a concurrent vector, it follows that

$$\underset{\xi}{\pounds} R^i_{jhk} = R^i_{jhk|a}\xi^a - 2R^i_{jhk},$$

$$\underset{\xi}{\pounds} P^i_{rj},{}^\alpha_a = g^{ib}(g_{dc}\gamma^d_{[b}\delta^s_{r]} + g_{c[b}\beta^s_{r]})(\underset{\xi}{\pounds}\Gamma^c_{sj})\ ;\ {}^\alpha_a,$$

$$\underset{\xi}{\pounds} S^i_{r},{}^\alpha_a,{}^\beta_b = 0,$$

where

$$R^i_{jhk} = \Gamma^i_{jh,\,k} - \Gamma^i_{jk,\,h} + \Gamma^i_{jh}\ ;\ {}^\alpha_a\Gamma^a_{\alpha k} - \Gamma^i_{jk}\ ;\ {}^\alpha_a\Gamma^a_{\alpha h} + \Gamma^a_{jh}\Gamma^i_{ak} - \Gamma^a_{jk}\Gamma^i_{ah},$$

$$P^i_{rj},{}^\alpha_a = \Gamma^i_{rj}\ ;\ {}^\alpha_a - C^i_{r},{}^\alpha_{a|j} + C^i_{r},{}^\beta_b p^c_\beta \Gamma^b_{cj}\ ;\ {}^\alpha_a,$$

$$S^i_{r},{}^\alpha_a,{}^\beta_b = C^i_{r},{}^\alpha_a\ ;\ {}^\beta_b - C^i_{c},{}^\alpha_a C^c_{r},{}^\beta_b - C^i_{r},{}^\beta_b\ ;\ {}^\alpha_a + C^i_{c},{}^\beta_b C^c_{r},{}^\alpha_a.$$

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