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On the Integrand Representation of Convex Operators

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凸作用素のインテグランド表現について

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Abstract

For a real valued convex function F, it is well known that $F^{**}=F$ holds if and only if F is lower semicontinuous (Fenchel-Moreau theorem). Moreover, if F is continuous at an interior point x of the domain of F, then F is subdifferentiable at x, that is $\partial F(x) \neq \emptyset$. For the generalized theory of convex operators with values in ordered vector spaces, we need the extensions of these facts. In [8] and other papers, some sufficient conditions for these facts are given. However, they require some strong assumptions concerning the relation between the order structure and the topological structure of the range of F. Our purpose is to give another approach to investigate convex operators by using convex integrands. This paper is a brief summary of the theory including some results in the case when F is defined on an infinite dimensional Banach space.

§1 INTRODUCTION

Let X, Y be real topological vector spaces and assume that Y is also an order complete vector lattice. An operator $F: X \longrightarrow Y$ is said to be convex if the domain of F(denoted by $D(F)$) is a convex set of X and

$$F(\lambda x+(1-\lambda)y) \leqq \lambda F(x)+(1-\lambda)F(y),$$

for all $x,y \in D(F)$ and $\lambda \in (0,1)$. By $L(X,Y)$, we denote the space of all continuous linear operators of X into Y. An element A of $L(X,Y)$ is called a subgradient of F at $x \in D(F)$ if $A(y-x) \leqq F(y)-F(x)$ for all $y \in D(F)$. The set of all subgradients of F at x is called the subdifferential of F at x and denoted by $\partial F(x)$. Furthermore, the conjugate operators F^* and F^{**} are defined as follows. For $A \in L(X,Y)$,

$$F^*(A)=\bigvee_{x\in D(F)}(A(x)-F(x)),$$

$$D(F^*)=\{A\in L(X,Y)\,|\,\bigvee_{x\in D(F)}(A(x)-F(x)) \text{ exists}\},$$

where $\vee$ denotes the lattice supremum in Y. Since X can be identified with a subspace of $L(L(X,Y),Y)$, we restrict the domain of F^{**} to X, and

$$F^{**}(x)=\bigvee_{A\in D(F^*)}(A(x)-F^*(A)),$$

$$D(F^{**})=\{x\in X\,|\,\bigvee_{A\in D(F^*)}(A(x)-F^*(A)) \text{ exists}\}.$$

F^* and F^{**} are also convex operators. Let (Ω,Σ,μ) be a σ-finite complete measure space and let $S(\Omega)$ be the space of finite valued measurable functions on Ω. f and g of $S(\Omega)$ are identified if they differ only on a set of μ-measure zero. With the usual ordering, $S(\Omega)$ is an order complete vector lattice. In most cases in applications, the range of convex operators can be regarded as subspaces of $S(\Omega)$ for some measure spaces Ω. Therefore it is not restrictive to consider only the class of convex operators with values in $S(\Omega)$.

A function $f: X\times\Omega \longrightarrow \mathrm{R}\cup\{+\infty\}$ is called a convex integrand, if for each $t\in\Omega$ the function $f(\cdot,t)$ is convex. A convex integrand f is said to be proper, if for every $t\in\Omega$, $f(\cdot,t)\not\equiv+\infty$. It is convenient to say that a proper convex integrand f has a constant domain, if $D(f(\cdot,t))$ does not depend on $t\in\Omega$ where $D(f(\cdot,t))=\{x\in X\,|\,f(x,t)<\infty\}$. If a proper convex integrand f has a constant domain and $f(x,\cdot)$ is measurable for each $x\in X$, the operator $F: X\longrightarrow S(\Omega)$ which acts according to the formula

$$(F(x))(t)=f(x,t),\quad (a.e.\, t\in\Omega), \tag{1}$$

is convex and $D(F)=D(f(x,\cdot))$. Conversely, if there exists a convex integrand f satisfying (1) for a convex operator F, we call f a representation of F.

§ 2 RESULTS IN FINITE DIMENSIONAL CASES

In this section, we consider only the case of $X=\mathrm{R}^d$. We have in this case the following fundamental theorem without any assumptions. ([3])

THEOREM 1. *Every convex operator* $F: \mathrm{R}^d\longrightarrow \mathrm{S}(\Omega)$ *has at least a representation.*

By the conjugate of the convex integrand f, we mean the integrand f^* on $X^*\times\Omega$ defined by

$$f^*(x^*, t) = \sup_{x \in D(f(\cdot,t))} \{<x^*, x> - f(x, t)\},$$

where X^* is the dual of X. The biconjugate integrand f^{**} is given by

$$f^{**}(x, t) = \sup_{x^* \in D(f^*(\cdot,t))} \{<x^*, x> - f^*(x^*, t)\},$$

for $x \in X$. If f is a proper convex integrand, then so are f^* and f^{**}. A proper convex integrand f is said to be normal if for each $t \in \Omega$, $f(\cdot, t)$ is lower semicontinuous and f is $\mathfrak{B} \otimes \Sigma$ measurable where $\mathfrak{B}$ denotes the σ-algebra of Borel subsets of X and $\mathfrak{B} \otimes \Sigma$ is the σ-algebra in $X \times \Omega$ generated by the sets $B \times S$ with $B \in \mathfrak{B}$ and $S \in \Sigma$. In [1], [5], [7], one can find some different ways to define the normality which are all equivalent. Normality ensures in particular that for every measurable function $x : \Omega \longrightarrow X$, the function $t \longrightarrow f(t, x(t))$ is also measurable, and it is extremely important in applications. Moreover, it is known that if f is normal then so are f^* and f^{**}. Note that for a convex operator F, the representation of F is not uniquely determined, and F does not always have a normal representation.

THEOREM 2. *If a convex integrand f represents a convex operator $F : \mathrm{R}^d \longrightarrow S(\Omega)$, then f^* and f^{**} are normal representations of F^* and F^{**} respectively.*

REMARK: $L(\mathrm{R}^d, S(\Omega))$ can be identified with $S(\Omega)^d$ by corresponding $(\varphi_1, \cdots, \varphi_d) \in S(\Omega)^d$ to $A \in L(\mathrm{R}^d, S(\Omega))$ with $A : \mathrm{R}^d \ni (x_1, \cdots, x_d) \longrightarrow \sum_{i=1}^{d} x_i \varphi_i \in S(\Omega)$. Hence we can consider R^d to be a subspace of $L(\mathrm{R}^d, S(\Omega))$, and the domain of F^* in Theorem 2 is restricted to R^d. For $\varphi = (\varphi_1, \cdots, \varphi_d) \in S(\Omega)^d$, we can also prove $(F^*(\varphi))(t) = f^*(\varphi(t), t)$ for almost every $t \in \Omega$.

PROOF: By the following lemma, we have

$$\begin{aligned} F^*(\varphi)(t) &= \bigvee_{x \in D(F)} (<\varphi, x> - F(x))(t) \\ &= \sup_{x \in D(F)} (<\varphi(t), x> - f(x, t)) \\ &= f^*(\varphi(t), t) \qquad (a.e. t \in \Omega), \end{aligned}$$

for every $\varphi \in S(\Omega)^d$. Hence we obtain the former statement by putting $\varphi(t) \equiv x^* \in \mathrm{R}^d$. Moreover, for every $x \in \mathrm{R}^d$,

$$\begin{aligned} F^{**}(x)(t) &= \bigvee_{\varphi \in S(\Omega)^d} (<\varphi, x> - F^*(\varphi))(t) \\ &\geqq \bigvee_{x^* \in \mathrm{R}^d} (<x^*, x> - F^*(x^*))(t) \end{aligned}$$

$$= \sup_{x^* \in \mathrm{R}^d} (<x^*, x> - f^*(x^*, t))$$

$$= f^{**}(x, t)$$

$$= \sup_{\varphi \in S(\Omega)^d} (<\varphi, x> - f^*(\varphi(t), t))$$

$$\geqq \bigvee_{\varphi \in S(\Omega)^d} (<\varphi, x> - F^*(\varphi))(t)$$

$$= F^{**}(x)(t) \qquad (a.e. t \in \Omega).$$

Hence $F^{**}(x)(t) = f^{**}(x, t)$, and this completes the proof.

LEMMA. *Let f be a representation of a convex operator $F : \mathrm{R}^d \longrightarrow S(\Omega)$, then*

$$(\bigwedge_{x \in D(F)} F(x))(t) = \inf_{x \in D(F)} f(x, t)$$

holds for a.e. $t \in \Omega$.

PROOF : Let E be a countable dense subset of $D(F)$. It is easy to see that

$$\inf_{x \in D(F)} f(x, t) = \inf_{x \in E} f(x, t)$$

for *a.e.* $t \in \Omega$. Hence $\inf_{x \in D(F)} f(x, t)$ is measurable in t and

$$(\bigwedge_{x \in D(F)} F(x))(t) \leqq (\bigwedge_{x \in E} F(x))(t)$$

$$= \inf_{x \in E} f(x, t)$$

$$= \inf_{x \in D(F)} f(x, t)$$

$$\leqq (\bigwedge_{x \in D(F)} F(x))(t)$$

for *a.e.* $t \in \Omega$, and the lemma has been proved.

By Theorem 2, we obtain the following two theorems.

THEOREM 3. *A convex operator $F : \mathrm{R}^d \longrightarrow S(\Omega)$ satisfies $F^{**} = F$ if and only if $f^{**} = f$ for some representation f.*

THEOREM 4. *A convex operator $F : \mathrm{R}^d \longrightarrow S(\Omega)$ satisfies $F^{**} = F$ if and only if F has a normal representation.*

To end this section, we show a generalization of the Fenchel-Moreau theorem. For a convex operator $F: \mathrm{R}^d \longrightarrow S(\Omega)$ and for $z \in D(F)$, denote

$S_F(z) = \{\varphi \in S(\Omega)_+ \mid F(U) \in F(z) - \varphi + S(\Omega)_+$, for some neighborhood U of z$\}$,

where $S(\Omega)_+ = \{\varphi \in S(\Omega) \mid \varphi(t) \geqq 0$ for almost every $t \in \Omega\}$.

THEOREM 5. *Let* $F: \mathrm{R}^d \longrightarrow S(\Omega)$ *be a convex operator and take a point* $x \in D(F)$. *Then* $F^{**}(x) = F(x)$ *if and only if* $S_F(x) \neq \emptyset$ and $\wedge\ S_F(x) = 0$.

The condition given in Theorem 5 is considered to be a generalization of the notion of lower semicontinuity for convex operators. A similar result and the proof can be seen in a preveous paper. Now by virtue of Theorem 2 and Theorem 3, it is quite easy to prove this theorem.

§ 3 INFINITE DIMENSIONAL CASES

Convex integrands on infinite dimensional spaces have been studied in many papers ([1] , [6]). The properties of normal convex integrands stated in § 2 are all valid in infinite dimensional cases. However, there are the usual problems of the multiplicity of topologies and dualities. We shall use some continuity conditions to prove an extension of Theorem 1.

THEOREM 6. *Let* X *be a separable reflexive Banach space and let* $F: X \longrightarrow S(\Omega)$ *be a convex operator. Suppose that* F *is continuous with respect to the almost everywhere convergence, that is,* $x_n \longrightarrow x$ *in* X *implies* $(F(x_n))(t) \longrightarrow (F(x))(t)$ *for almost every* $t \in \Omega$. *Then* F *has a normal representation.*

PROOF : Let E be a countable dense subset of X. We can assume that E is midpoint convex, that is, $x, y \in E$ implies $\frac{1}{2}(x+y) \in E$. By D, we denote the set of all rational numbers of the form $\lambda = \frac{n}{2^m} \in [0,\ 1]$. For each $x, y \in E$ *and* $\lambda \in D$, $\lambda x + (1-\lambda) y$ belongs to E and by the convexity of F,

$$(F(\lambda x + (1-\lambda)y))(t) \leqq \lambda (F(x))(t) + (1-\lambda)(F(y))(t) \tag{2}$$

holds for all $t \in \Omega \setminus \Omega_1(x,y,\lambda)$ where $\Omega_1(x,y,\lambda) \subset \Omega$ is μ-measure zero. Take the union of $\Omega_1(x,y,\lambda)$ over all $x,y \in E$ and $\lambda \in D$, and denote it by Ω_2. Then $\mu(\Omega_2) = 0$ and(2)holds on $\Omega \setminus \Omega_2$ for all $x,y \in E$ and $\lambda \in D$. Hence if we define $f(x,t)$ on $E \times \Omega$ by $f(x,t) = (F(x))(t)$ for $x,y \in E$ and $t \in \Omega$, then f satisfies

$$f(\lambda x + (1-\lambda)y, t) \leqq \lambda f(x,t) + (1-\lambda) f(y,t) \tag{3}$$

for all $x, y \in E$, $\lambda \in D$, and $t \in \Omega \setminus \Omega_2$. For every $x \in X$, $t \in \Omega \setminus \tilde{\Omega}(x)$, $(\mu(\tilde{\Omega}(x)) = 0)$, and $\varepsilon > 0$, there exists $\delta = \delta(x,t,\varepsilon) > 0$ such that $\| x - y \| < \delta$ implies $| F(x))(t) - (F(y))(t) | < \varepsilon$, by the continuity condition of F. Hence for each $t \in \Omega \setminus (\Omega_2 \cup \tilde{\Omega}(x))$,

$$| (F(x))(t) - f(y,t) | < \varepsilon$$

holds for all $y \in E \cap V_\delta(x)$, where $V_\delta(x)$ denotes the δ-neighborhood of x. Hence for each $t \in \Omega \setminus (\Omega_2 \cup \tilde{\Omega}(x))$, $f(\cdot, t)$ is bounded on $E \cap V_\delta(x)$, and by (3), this implies the uniform continuity of $f(\cdot, t)$ on $E \cap V_\delta(x)$. Thus we can define $f(x,t)$ on $X \times \Omega$ by the usual way of taking the limit. f is obviously a convex integrand, and also normal since $f(\cdot, t)$ is continuous on X. Again by the continuity condition of F, we obtain, for each $x \in X$,

$$\begin{aligned} (F(x))(t) &= \lim_{n\to\infty}(F(x_n))(t) \\ &= \lim_{n\to\infty} f(x_n, t) \\ &= f(x,t) \end{aligned}$$

for almost every $t \in \Omega$, where $\{x_n\}$ is a sequence of E converging to x. Thus f is a normal representation of F, and this completes the proof.

Let $\mathfrak{F}(X,\Omega)$ denote the set of all convex operators $F : X \longrightarrow S(\Omega)$ satisfying the continuity condition in Theorem 6. In the infinite dimensional cases, we note that the definitions of F^* and F^{**} depend on the meaning of $L(X,S(\Omega))$. Under the hypothesis in Theorem 6, we define $L(X,S(\Omega))$ as the set of all linear operators satisfying the continuity condition in Theorem 6. Then we can obtain the following result.

THEOREM 7. *Let X be a separable reflexive Banach space. Then Theorem 3 and Theorem 4 remain valid for $F \in \mathfrak{F}(X, \Omega)$.*

Since the representation f obtained in Theorem 6 is normal, we have by Theorem 7 that

THEOREM 8. *Let X be a separable reflexive Banach space. If $F \in \mathfrak{F}(X,\Omega)$, then F satisfies $F^{**} = F$.*

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