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Infinitesimal Projective Transformations on the Tangent Bundles with the Horizontal Lift Connection

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Abstract

Let (M, g) be a complete Riemannian manifold and TM its tangent bundle with the horizontal lift connection. If TM admits a non-affine infinitesimal projective transformation, then M and TM are locally flat.

1. Introduction

In the present paper everything will be always discussed in the C^∞-category, and manifolds will be assumed to be connected and dimension $n > 1$.

Let M be a manifold and TM its tangent bundle. We denote by $\mathfrak{T}^r_s(M)$ the set of all tensor fields of type (r, s) on M. Similarly, we denote by $\mathfrak{T}^r_s(TM)$ the corresponding set on TM.

Let ∇ be an affine connection on M. A vector field V on M is called an infinitesimal projective transformation if there exists a 1-form Ω such that

$$(L_V\nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X$$

for any $X, Y \in \mathfrak{T}^1_0(M)$, where L_V is the Lie derivation with respect to V. In this case Ω is called the associated 1-form of V. Especially, if $\Omega = 0$, then V is called an infinitesimal affine transformation.

Next, let g be a Riemannian metric on M and ∇ the Levi-Civita connection of g. Denote by V a vector field on M. V is called an infinitesimal conformal transformation if there exists a function f on M satisfying $L_V g = fg$. Especially, if f is constant, then V is called an infinitesimal homothetic transformation. Furthermore, if $f = 0$, then V is called an infinitesimal isometry .

In this paper, we prove the following

Theorem. *Let (M, g) be a complete Riemannian manifold and TM its tangent bundle with the horizontal lift connection. If TM admits a non-affine infinitesimal projective transformation, then M and TM* are *locally flat.*

2. Preliminaries

Let (M, g) be a Riemannian manifold, ∇ the Riemannian connection of g and $\Gamma_{ji}{}^{h}$ the coefficients of ∇ , i.e., $\Gamma_{ji}{}^{a}\partial_a := \nabla_{\partial_j}\partial_i$, where $\partial_h = \dfrac{\partial}{\partial x^h}$ and (x^h) is the local coordinates of M. We define a local frame $\{E_i , E_{\bar{i}}\}$ of TM as follows :

$$E_i := \partial_i - y^b \Gamma_{ib}{}^{a}\partial_{\bar{a}} \text{ and } E_{\bar{i}} := \partial_{\bar{i}},$$

where (x^h, y^h) is the induced coordinates of TM and $\partial_{\bar{i}} := \dfrac{\partial}{\partial y^i}$, and call this frame $\{E_i , E_{\bar{i}}\}$ the adapted frame of TM. Then $\{dx^h, \delta y^h\}$ is the dual frame of $\{E_i , E_{\bar{i}}\}$, where $\delta y^h := dy^h + y^b \Gamma_{ab}{}^{h} dx^a$.

By the definition of the adapted frame, we have the following

Lemma 1 . *The Lie brackets of the adapted frame of TM satisfy the following identities :*

(1) $[E_j , E_i] = y^b K_{ijb}{}^{a} E_{\bar{a}}$

(2) $[E_j , E_{\bar{i}}] = \Gamma_{ji}{}^{a} E_{\bar{a}}$,

(3) $[E_{\bar{j}}, E_{\bar{i}}] = 0$,

where $K = (K_{kji}{}^{h})$ denotes the Riemannian curvature tensor of (M, g) defined by $K_{kji}{}^{h} := \partial_k \Gamma_{ji}{}^{h} - \partial_j \Gamma_{ki}{}^{h} + \Gamma_{ji}{}^{a}\Gamma_{ka}{}^{h} - \Gamma_{ki}{}^{a}\Gamma_{ja}{}^{h}$.

Let $\tilde{\nabla}$ be the horizontal lift connection on TM defined as follows :

$$\tilde{\nabla}_{E_j} E_i = \Gamma_{ji}{}^{a} E_a , \ \tilde{\nabla}_{E_j} E_{\bar{i}} = \Gamma_{ji}{}^{a} E_{\bar{a}} ,$$
$$\tilde{\nabla}_{E_{\bar{j}}} E_i = 0 , \ \tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0.$$

This connection is the metric connection of the comlete lift metric $\tilde{g} = 2g_{ba} dx^b \delta y^a$ or the lift metric $\tilde{g} = g_{ba} dx^b dx^a + 2g_{ba} dx^b \delta x^a$. But it is not necessary for the present paper to use the lift metric itself.

We need the following well known lemma to prove Theorem.

Lemma 2 ([K1]). *If a complete Riemannian manifold M admits a non-isometric infinitesimal homothetic transformation, then M is locally flat.*

3. Infinitesimal projective transformation on *TM*

Proposition 1 . *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the horizontal lift connection. $\tilde{V}$ is an infinitesimal projective transformation with the associated 1-form $\tilde{\Omega}$ on TM if and only if there exist $\varphi, \psi \in \Im_0^0(M)$, $B = (B^h)$, $D = (D^h) \in \Im_0^1(M)$, $A = (A_i^h)$, $C = (C_i^h) \in \Im_1^1(M)$ satisfying*

(1) $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (B^h + y^a A_a^h, D^h + y^a C_a^h + y^a y^h \Phi_a)$,

(2) $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\partial_i \psi, \partial_i \varphi) = (\Psi_i, \Phi_i)$,

(3) $\nabla_j \Phi_i = 0$, $\nabla_j \Psi_i = 0$,

(4) $\nabla_j A_i^h = \Phi_i \delta_j^h$,

(5) $\nabla_j C_i^h = \Psi_j \delta_i^h - K_{aji}{}^h B^a$,

(6) $L_B \Gamma_{ji}{}^h = \nabla_j \nabla_i B^h + K_{aji}{}^h B^a = \Psi_j \delta_i^h + \Psi_i \delta_j^h$,

(7) $\nabla_j \nabla_i D^h = 0$.

(8) $K_{kja}{}^h A_i^a = 0$,

where $(\tilde{V}^h, \tilde{V}^{\bar{h}}) := \tilde{V}^a E_a + \tilde{V}^{\bar{a}} E_{\bar{a}} = \tilde{V}$, $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) := \tilde{\Omega}_a dx^a + \tilde{\Omega}_{\bar{a}} \delta y^a = \tilde{\Omega}$.

Proof. Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let $\tilde{V}$ be an infinitesimal projective transformation with the associated 1-form $\tilde{\Omega}$ on TM.

From $(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = \tilde{\Omega}_{\bar{j}} E_{\bar{i}} + \tilde{\Omega}_{\bar{i}} E_{\bar{j}}$, we obtain

(3.1) $$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^h = 0$$

and

(3.2) $$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} = \tilde{\Omega}_{\bar{j}} \delta_i^h + \tilde{\Omega}_{\bar{i}} \delta_j^h .$$

From (3.1), we have

$$\partial_{\bar{i}} \tilde{V}^h = A_i^h$$

and

(3.3) $$\tilde{V}^h = B^h + y^a A_a^h ,$$

where B^h and A_i^h are certain functions which depend only on x^h. The coordinate transformation rule implies that $B = (B^h) \in \Im_0^1(M)$ and $A = (A_i^h) \in \Im_1^1(M)$.

From $(L_{\tilde{V}} \tilde{\nabla})(E_{\bar{j}}, E_i) = \tilde{\Omega}_{\bar{j}} E_i + \tilde{\Omega}_i E_{\bar{j}}$, using (3.3), we obtain

(3.4) $$\tilde{\Omega}_{\bar{j}} \delta_i^h = \nabla_i A_j^h$$

and

(3.5) $$\tilde{\Omega}_j \delta_i^h = K_{aij}{}^h B^a + y^a \left(K_{bij}{}^h A_a^b + K_{bia}{}^h A_j^b \right) + \Gamma_{ai}{}^h \partial_{\bar{j}} \tilde{V}^{\bar{a}} \partial_{\bar{j}} \left(E_i \tilde{V}^{\bar{h}} \right).$$

Contracting j and h in (3.4), we have $\tilde{\Omega}_{\bar{j}} = \nabla_i A_a^a$ · Therefore we have

(3.6) $$\tilde{\Omega}_{\bar{i}} = \Phi_i = \partial_i \varphi,$$

where $\varphi_i := A_a^a$ and $\Phi_i := \partial_i \varphi$. From (3.4) and (3.6), we get

(3.7) $$\nabla_j A_i^h = \Phi_i \delta_j^h .$$

From (3.2) and (3.6), we have

$$\partial_{\bar{i}} \tilde{V}^{\bar{h}} = C_i^h + y^h \Phi_i + y^a \Phi_a \delta_i^h$$

and

(3.8) $$\tilde{V}^{\bar{h}} = D^h + y^a C_a^h + y^a y^h \Phi_a ,$$

where D^h and $C_i{}^h$ are certain functions which depend only on x^h. We can see that $D = (D^h) \in \Im_0^1(M)$ and $C = (C_i{}^h) \in \Im_1^1(M)$.

Substituting (3.8) into (3.5), we obtain

(3.9) $$\tilde{\Omega}_i \delta_j^h = \nabla_i C_j{}^h + K_{aij}{}^h B^a + y^a \left(K_{bij}{}^h A_a^b + K_{bia}{}^h A_j^b + \delta_a^h \nabla_i \Phi_j + \delta_j^h \nabla_i \Phi_a \right).$$

On the other hand, from $(L_{\tilde{V}} \tilde{\nabla})(E_j, E_{\bar{i}}) = \tilde{\Omega}_j E_{\bar{i}} + \tilde{\Omega}_{\bar{i}} E_j$, we get

(3.10) $$\tilde{\Omega}_j \delta_i^h = \nabla_i C_j^h + K_{aji}{}^h B^a + y^a \left(K_{bji}{}^h A_a^b + \delta_a^h \nabla_j \Phi_i + \delta_i^h \nabla_j \Phi_a \right).$$

Comparing (3.9) with (3.10), we obtain

(3.11) $$K_{kja}{}^{h} A_i^a = 0$$

and

(3.12) $$\tilde{\Omega}_j \delta_i^h = \nabla_j C_i^h + K_{aji}{}^{h} B^a + y^a \left(\delta_a^h \nabla_j \Phi_i + \delta_i^h \nabla_j \Phi_a \right).$$

Contracting i and h in (3.12), we get

(3.13) $$n\tilde{\Omega}_j = \nabla_j C_a^a + (n+1) y^a \nabla_j \Phi_a .$$

Here, we put $\psi := \frac{1}{n} C_a^a$ and $\Psi_i := \nabla_i \psi = \partial_i \psi$. Then, from (3.12) and (3.13), we obtain

(3.14) $$\tilde{\Omega}_i = \Psi_i + \frac{n+1}{n} y^a \nabla_i \Phi_a ,$$

(3.15) $$\nabla_j C_i{}^{h} = \Psi_j \delta_i^h - K_{aji}{}^{h} B^a ,$$

and

(3.16) $$n \delta_k^h \nabla_j \Phi_i - \delta_i^h \nabla_j \Phi_k = 0.$$

Contracting k and h in (3.16), we have

(3.17) $$\nabla_j \Phi_i = 0.$$

From (3.14) and (3.17), we get

(3.18) $$\tilde{\Omega}_i = \Psi_i .$$

From $(L_{\tilde{V}} \tilde{\nabla})(E_j, E_i) = \tilde{\Omega}_j E_i + \tilde{\Omega}_i E_j$, we obtain

(3.19) $$L_B \Gamma_{ji}{}^{h} = \Psi_j \delta_i^h + \Psi_i \delta_j^h ,$$

(3.20) $$\nabla_j \nabla_i D^h = 0$$

and

(3.21) $$\nabla_k \nabla_j C_i{}^{h} + \nabla_k \left(K_{aji}{}^{h} B^a \right) = 0.$$

Substituting (3.15) into (3.21), we have

(3.22) $$\nabla_j \Psi_i = 0.$$

Q.E.D.

Proof of Theorem.

We put $X^h := A_a{}^{h} \Phi^a$. Then, using (3.7) and (3.17), we have

(3.23) $$\begin{aligned} L_X g_{ji} &= \nabla_j X_i + \nabla_i X_j \\ &= (\nabla_j A_{ai}) \Phi^a + (\nabla_i A_{aj}) \Phi^a \\ &= 2(\Phi_a \Phi^a) g_{ji} . \end{aligned}$$

Similarly we put $Y^h := \left(\nabla_a B^h - C_a{}^{h} \right) \Psi^a$. Then, using (3.15), (3.19) and (3.22),

(3.24) $$\begin{aligned} L_Y g_{ji} &= (\nabla_j \nabla_a B_i - \nabla_j C_{ai}) \Psi^a + (\nabla_i \nabla_a B_j - \nabla_i C_{aj}) \Psi^a \\ &= \left\{ \left(-K_{bjai} B^b + \Psi_j g_{ai} + \Psi_a g_{ji} \right) - \left(\Psi_j g_{ai} - K_{bjai} B^b \right) \right\} \Psi^a + (\Psi_a g_{ji}) \Psi^a \\ &= 2(\Psi_a \Psi^a) g_{ji} . \end{aligned}$$

Therefore X and Y are infinitesimal homothetic transformations. If M is not locally flat, then $\Phi = \Psi = 0$ by virtue of Lemma 2, and consequently $\tilde{\Omega} = 0$. This is a contradiction. Therefore M is locally flat. In this case TM is also locally flat.

Q.E.D.

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