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A Diagrammatic Interpretation of Fitzgerald-Lorentz Contraction in Special Relativity

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特殊相対性理論における フィッツジェラルド-ローレンツ収縮の図形的解釈

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ABSTRACT

A diagrammatic interpretation of Fitzgerald-Lorentz contraction in special relativity is given here based on the well-known diagram obtained from the Michelson-Morley experiment. This interpretation is the first of its kind as far as the author knows. Also outlined is a simple method of depicting the contracted length on a diagram based on the interpretation, and proof is given using both geometrical and analytical methods.

1 Introduction

Introducing various ways of representing the laws of physics is expected to lead students to a better understanding of nature, and Fitzgerald-Lorentz contraction in special relativity is one such scientific consideration. This contraction is given from a purely logical way of thinking based on assumptions such as the laws of physics being identical in all inertial frames. As it is purely logical, students sometimes find it hard to visualize its meaning. Accordingly, the purpose of the present paper is to give a diagrammatic interpretation of the contraction based on the well-known Michelson-Morley experiment. This will, the author believes, help students to better understand its physical meaning.

The next section gives a brief review of how the contraction is derived from a purely logical way of thinking based on an analytical method. In the third section, a diagrammatic interpretation of the contraction is given based on the diagram from the Michelson-Morley experiment. In the fourth section, a simple method of depicting the length of contraction on the diagram is presented. To end, some concluding remarks are made in the final section.

2 Fitzgerald-Lorentz Contraction using an Analytical Method

Before discussion of the diagrammatic interpretation of Fitzgerald-Lorentz contraction, the analytical method that leads to Lorentz transformation will be briefly reviewed.

Consider two inertial frames, S and S', and assume that S' is moving with a constant speed v in the direction of the positive x axis of S. We also assume that time is calibrated for these frames by setting $t = t' = 0$ for the moment when the origins of the two frames coincide. The transformation of the x and t coordinates of these frames would take the forms[1)] of

$$x' = ax - bt; \qquad x = ax' + bt', \tag{1}$$

as reduction to the Galilean transformation should be seen

$$x' = x - vt; \qquad x = x' + vt' \tag{2}$$

when v is much smaller than the speed c of light. In Eq. (1), a and b are parameters to be determined, and b is readily reducible to

$$b = va, \tag{3}$$

because the origin of S' always has the coordinate $x' = 0$, and at time t we can write

$$0 = ax - bt; \quad \Rightarrow \quad \frac{x}{t} = \frac{b}{a}, \tag{4}$$

and x/t is simply the speed v of the origin of S' in S:

$$v = \frac{b}{a}. \tag{5}$$

To determine a, we can use the property of c being the same in all inertial frames, which is reducible[2] from the postulate that the laws of physics are identical in all inertial frames. If a light signal is emitted from the origin at $t = t' = 0$, it will travel the distance

$$x = ct, \quad \text{and} \quad x' = ct'. \tag{6}$$

Substituting Eqs. (3) and (6) into the first equality of Eq. (1) gives

$$ct' = act - avt, \quad \Rightarrow \quad \frac{t'}{t} = a\left(1 - \frac{v}{c}\right). \tag{7}$$

Similarly, substitution into the second equality of Eq. (1) gives

$$ct = act' + avt', \quad \Rightarrow \quad \frac{t}{t'} = a\left(1 + \frac{v}{c}\right). \tag{8}$$

Multiplying both sides of Eqs. (7) and (8) gives

$$1 = a^2\left(1 - \frac{v^2}{c^2}\right), \quad \Rightarrow \quad a = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma, \tag{9}$$

where γ is known as the Lorentz factor. The transformation eventually takes the form of

$$x' = \gamma(x - vt); \qquad x = \gamma(x' + vt'). \tag{10}$$

Now we look at Fitzgerald-Lorentz contraction. We consider a rod on the x' axis at rest in S' and moving with speed v in S. The length of the rod in S' is the difference in the x' coordinates of the edges

$$L_{\text{at rest}} = x_2' - x_1', \tag{11}$$

where the subscript "at rest" means that the rod is at rest in the frame S'. Substituting the transformation of Eq. (10) into Eq. (11) gives

$$L_{\text{at rest}} = \gamma(x_2 - vt) - \gamma(x_1 - vt) = \gamma(x_2 - x_1). \tag{12}$$

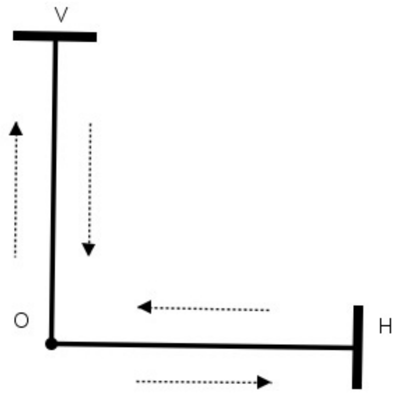

Figure 1: Schematic diagram of the Michelson-Morley experiment. A light source is placed at O, and two mirrors are placed at H and V. The distance from O and the mirrors is L ($|\overline{OH}| = |\overline{OV}| = L$), so if the apparatus is at rest the light signals emitted simultaneously at the origin O will return to O simultaneously. (All the figures in this paper were drawn using a free software GEONExT[3].)

The last factor $(x_2 - x_1)$ is the length of the moving rod in S so that the equality reduces to

$$L_{\text{at rest}} = \gamma L_{\text{moving}}, \tag{13}$$

or

$$L_{\text{moving}} = \frac{L_{\text{at rest}}}{\gamma} = L_{\text{at rest}}\sqrt{1 - \frac{v^2}{c^2}}. \tag{14}$$

The moving rod reduces its length by the factor of $1/\gamma$.

3 Fitzgerald-Lorentz Contraction using a Diagrammatic Method

We start with the well-known experiment performed by Michelson and Morley, whose apparatus is shown in Fig. 1. It is comprised of a light source at O and two mirrors at H and V placed an equal distance L from O. If the apparatus is at rest, light signals emitted at O will travel to the mirrors, where they will be reflected and return to the light source O simultaneously.

However, if the apparatus is moving with speed v in the positive x direction, the light signal traveling in the "vertical" direction will take the pathway from O to B and from B to O' as shown in Fig. 2. It can be seen that the pathway from O to B and that from B to O' have the same length

$$|\overline{OB}| = |\overline{BO'}|, \tag{15}$$

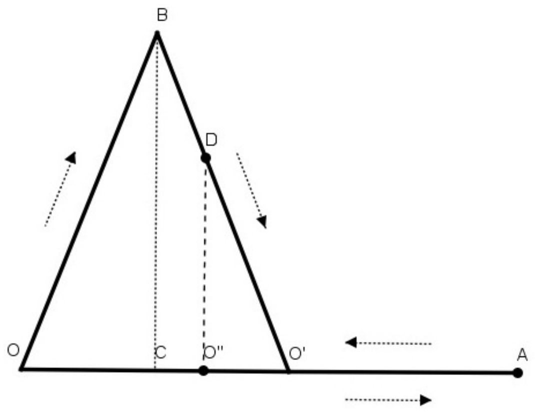

Figure 2: Paths of light signals traveling in the vertical and horizontal directions in the Michelson-Morley experiment with moving apparatus. The distance of the vertical path is the same as that of the horizontal path: $|\overline{OB}| + |\overline{BO'}| = |\overline{OA}| + |\overline{AO'}|$, which means $|\overline{OB}| = |\overline{CA}|$ since $|\overline{OA}| = 2 \times |\overline{CO'}| + |\overline{AO'}|$. The triangle $\triangle BOO'$ is an isosceles triangle, and the point C is the center of the base $\overline{OO'}$. The moment the horizontal light signal arrives at A, the vertical light signal arrives at a point (denoted as D) the same distance $|\overline{AO'}|$ away from O'. The light source should therefore be at O'', since it is always right below the vertical light signal.

so the triangle $\triangle BOO'$ is an isosceles triangle. The point C is the center of the base $\overline{OO'}$, and the angle $\angle C$ becomes a right angle.

The light signal traveling in the "horizontal" direction will take the pathway from O to A and then from A to O', as shown in Fig. 2. The light signals starting at O in both the vertical and horizontal directions will arrive at O' simultaneously, which means that the lengths of these pathways are the same:

$$|\overline{OB}| + |\overline{BO'}| = |\overline{OA}| + |\overline{AO'}|. \tag{16}$$

Since the segment $\overline{OA}$ decomposes into three segments

$$\overline{OA} = \overline{OC} + \overline{CO'} + \overline{O'A}, \tag{17}$$

we can prove that the segment $\overline{CA}$ has the same length as $\overline{BO}$:

$$|\overline{CA}| = |\overline{CO'}| + |\overline{O'A}| = |\overline{BO}|\,. \tag{18}$$

Now we look at the horizontal distance, which in Fig. 1 is the length between the light source O and the horizontal mirror at H. For this purpose, we consider the position of the light source at the moment when the horizontal light signal arrives at the horizontal mirror H. The position of the mirror is at A in Fig. 2, since the signal is reflected there. How about the position of the light source O at that moment? Before considering its position, it should be noted that the horizontal light signal remains at the distance $|\overline{AO'}|$ for the full journey to O' in Fig. 2. This means that the light signal in the "vertical" direction also remains at the distance $|\overline{AO'}|$. It should also be noted that the position of the light source O is always right below the vertical light signal. If we plot a point D on the line segment $\overline{BO'}$ with

$$|\overline{DO'}| = |\overline{AO'}|, \tag{19}$$

it denotes the position of the vertical light signal at the moment when the horizontal light signal arrives at A. Then, the position of the light source becomes O'', obtained by drawing a vertical line right below D. Thus, the desired positions are found, and we can depict the length of Fitzgerald-Lorentz contraction by connecting these positions, namely, the length of the line segment $\overline{AO''}$:

$$L_{\text{moving}} = |\overline{AO''}|. \tag{20}$$

Let us prove that the distance $|\overline{AO''}|$ is the same as that in Eq. (14) derived in the previous section. First of all, let us denote the elapsed time τ of the light going from O to B in Fig. 2. Within this elapsed time τ, the apparatus moves from O to C. The length $|\overline{BC}|$ is, of course, L. These relations can be put into the following equalities:

$$\begin{aligned} |\overline{OB}| &= c\tau, \\ |\overline{OC}| &= v\tau, \\ |\overline{CB}| &= L. \end{aligned} \tag{21}$$

The value of τ is readily obtained using the Pythagorean theorem for the right-side triangle $\triangle BCO$:

$$(c\tau)^2 = (v\tau)^2 + L^2, \quad \Rightarrow \quad \tau = \frac{L}{\sqrt{c^2 - v^2}}. \tag{22}$$

The distance $|\overline{AO'}|$ is the difference between $|\overline{CA}|$ and $|\overline{CO'}|$. It should also be noted that $|\overline{CA}| = |\overline{OB}|$:

$$|\overline{AO'}| = |\overline{CA}| - |\overline{CO'}| = c\tau - v\tau. \tag{23}$$

The right-side triangles $\triangle BCO'$ and $\triangle DO''O'$ are apparently similar, and $|\overline{DO'}| = |\overline{AO'}|$, so the ratio

$$|\overline{O''O'}| : |\overline{DO'}| = |\overline{CO'}| : |\overline{BO'}| \tag{24}$$

gives the distance $|\overline{O''O'}|$ as

$$|\overline{O''O'}| = \frac{v}{c}(c - v)\tau. \tag{25}$$

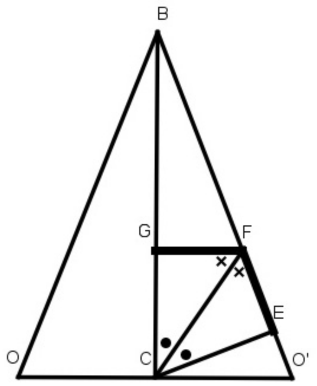

Figure 3: A geometric proof of the contracted length L/γ. The right-side triangle $\triangle BFG$ is set to be congruent with the right-side triangle $\triangle DO'O''$ in Fig. 2. The right-side triangles $\triangle FGC$ and $\triangle FEC$ are found to be congruent with each other since they have the same angles [denoted by crosses ("x") and dots ("•")] and a common hypotenuse. Then, we obtain the relation $|\overline{FG}| = |\overline{FE}|$, and finally get $|\overline{BE}| = |\overline{AO''}|$.

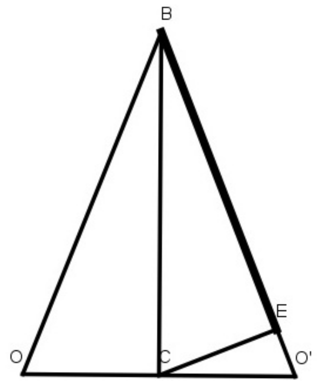

Figure 4: The contracted length L/γ is depicted by the thick line. From the point C (the center of the base $\overline{OO'}$) a line is drawn orthogonal to the segment $\overline{BO'}$. The crossing point is denoted as E, and the segment $\overline{BE}$ thus has the length L/γ.

From these results, we evaluate the distance $|\overline{AO''}|$ as

$$|\overline{AO''}| = |\overline{O''O'}| + |\overline{AO'}| = L\sqrt{1-\frac{v^2}{c^2}} = \frac{L}{\gamma}. \quad (26)$$

This means that $|\overline{AO''}|$ is the same as the L_{moving} value given by Eq. (14).

4 Contracted Length in Diagram

It is interesting that the length $|\overline{AO''}|$ in Fig. 2 is readily depicted on the segment $\overline{BO'}$ of that figure. If we plot a line from the center C of the base $\overline{OO'}$ of the isosceles triangle $\triangle BOO'$ that falls perpendicular to the leg $\overline{BO'}$, and call the crossing point E (see Fig. 3), the line segment $\overline{BE}$ has a length equal to that of $\overline{AO''}$. We can prove this equality in two ways, one a geometric method and the other an analytical method.

First, the geometric proof will be given. For this purpose, the right-side triangle $\triangle DO'O''$ in Fig. 2 is moved upward along the segment $\overline{BO'}$ until D coincides with B. The new positions of the points O' and O'' are denoted as F and G, respectively, as shown in Fig. 3. As has been noted, $|\overline{CA}| = |\overline{BO'}|$ and $|\overline{AO'}| = |\overline{DO'}| = |\overline{BF}|$, so we have

$$|\overline{CO'}| = |\overline{FO'}|. \quad (27)$$

This means that the triangle $\triangle O'CF$ is an isosceles triangle, and so the base angles $\angle O'CF$ and $\angle O'FC$ have the same measure:

$$\angle O'CF = \angle O'FC. \quad (28)$$

The two line segments $\overline{CO'}$ and $\overline{GF}$ are parallel, so the alternate interior angles $\angle O'CF$ and $\angle CFG$ also have the same measure:

$$\angle O'CF = \angle CFG. \quad (29)$$

We therefore have

$$\angle CFO' = \angle CFE = \angle CFG, \quad (30)$$

which is denoted by the crosses ("x") in Fig. 3. The angles at E and G are right angles, and the remaining angles also have the same measure:

$$\angle ECF = \angle GCF \quad (31)$$

as denoted by the dots ("•") in the figure. The right-side triangles $\triangle CEF$ and $\triangle CGF$ become congruent since they have a common hypotenuse and two equal angles (that is, two angles and the included side). Then we have

$$|\overline{FE}| = |\overline{FG}|. \quad (32)$$

Noting that $|\overline{BF}| = |\overline{DO'}| = |\overline{AO'}|$ and $|\overline{FE}| = |\overline{FG}| = |\overline{O'O''}|$, we finally obtain the relation

$$|\overline{BE}| = |\overline{BF}| + |\overline{FE}| = |\overline{AO'}| + |\overline{O'O''}| = |\overline{AO''}|. \quad (33)$$

An analytical proof is as follows: the right-side triangles $\triangle BOC$ and $\triangle BCE$ are apparently similar, as

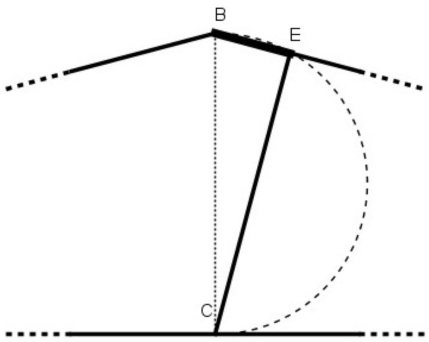

Figure 5: The contracted length L/γ becomes shorter as the speed v increases since the triangle becomes flatter as v becomes larger, so that the orthogonal line from C crosses at a point nearer to B. The point E is actually a point on a semicircle with a diameter of $|\overline{BC}|$.

can be seen in Fig. 4. Then, we use the scale factors of similar triangles:

$$|\overline{BO}| : |\overline{BC}| = |\overline{BC}| : |\overline{BE}|. \tag{34}$$

Substituting Eqs. (21), (22) and (9), we readily obtain

$$|\overline{BE}| = L\sqrt{1 - \frac{v^2}{c^2}} = \frac{L}{\gamma}. \tag{35}$$

Before ending this section, we note that the representation of the contracted length on the leg $\overline{BO'}$ provides a simple method indicating that faster values of v correspond to shorter reductions of Fitzgerald-Lorentz contraction, as shown in Fig. 5. The isosceles triangle $\triangle BOO'$ in Fig. 4 becomes flatter as v becomes larger, so the orthogonal line from C crosses at a point nearer to B. The point E is actually a point on a semicircle with a diameter of $|\overline{BC}|$ because the angle $\angle BEC$ is always a right angle (see Fig. 5). The figure also shows that the length becomes zero when the value of v reaches c.

$$|\overline{BE}| \to 0 \quad \text{as} \quad v \to c. \tag{36}$$

5 Concluding Remarks

In this paper, a diagrammatic interpretation of Fitzgerald-Lorentz contraction was given based on the diagram obtained from the Michelson-Morley experiment.

The contracted length is depicted by investigating the position of the light source at the moment when the horizontal light signal arrives at the horizontal mirror. This is found as the length of the line segment $\overline{AO''}$ in Fig. 2.

A simple method of depicting the contracted length on this diagram was also given by drawing an orthogonal line from the center C of the base of the isosceles triangle $\triangle BOO'$ to the line segment $\overline{BO'}$ as shown in Fig. 4. The contracted length is found to be the length of the line segment $\overline{BE}$.

From the length of the line segment $\overline{BE}$ in Fig. 5, it is easy to understand that the contracted length becomes smaller as v approaches c, and that it takes a value of zero when $v = c$.

The interpretation and depiction given in this paper are the first of their kind as far as the author knows, although various derivations based on analytical methods have been given by many researchers and textbook authors.

Students who have previously struggled to visualize the contraction will be able to easily create a pictorial image of it based on the diagram given in this paper and, the author expects, thereby develop a better understanding of the concept of special relativity.

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