負数次のヘルダー総和法について

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On the Hölder Mean of Negative Order (I)

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三浦白治： 負数次のヘルダー総和法について

1. Introduction. Amnon Jakimovski and C. T. Rajagopal proved the following Tauberian theorem for the Hölder mean of negative order $-k$ in [1], [2].

Theorem of Jakimovski. Let k be a fixed positive integer. A necessary and sufficient condition for $\{s_n\}$ to be summable $(H, -k)$ to s is that $\{s_n\}$ should be summable (A) to s and

$$\lim_{n\to\infty}\binom{n}{k}\cdot\Delta^k s_{n-k}=0.$$

Theorem of C. T. Rajagopal. (a) If (i) $\{s_n\}$ is summable (A) to s, and if (ii) for a positive integer k, $n^k\Delta^k s_{n-k}=0$, $n\to\infty$, then $\{s_n\}$ is summable $(H, -k)$ to s.
(b) Conditions (i) and (ii) are also necessary for $\{s_n\}$ to be summable $(H, -k)$ to s.

In this paper, we will extend the above result to the case of some fixed positive fractional number k.

2. An extension of the above theorem. By the application of the difference of fractional order defined in [3], we will make an extension of the above theorems in the case of a positive fractional number k.

Theorem 1. If $\{s_n\}$ is summable (A) to s and if, for a fractional number k, $0<k<1$,

$$n^k S^{-k}(s_n)^{1)} = 0(1),\quad n\to\infty,$$

then $\{s_n\}$ is summable $(H, -k)$ to s, where $s_n=a_0+a_1+\cdots+a_n$.

Proof. Let us now put $k=-\beta$. From our assumption $0<k<1$, we have $-1<\beta<0$. Then our hypothesis is that

$$n^{-\beta}S^{\beta}(s_n)=0(1),\quad n\to\infty\ (\beta>-1),$$

that is, $$(S^{\beta}(s_n)/A_n^{\beta})\cdot(A_n^{\beta}/n^{\beta})=0(1),\quad n\to\infty.$$

1) The difference of fractional order in [3] is defined by the operation of "inverse summation":

$$S^{-k}(s_n)=\sum_{\nu=0}^{n}A_{n-\nu}^{-k-1}s_\nu,$$

which reduces, in the case $k=1$, to

$$S^{-1}(s_n)=s_n-s_{n-1},$$

provided we write $s_{-1}=0$. Writing $S^k(s_n)=\sum_{\nu=0}^{n}A_{n-\nu}^{k-1}s_\nu$, we have, for all real α, β, $S^{\alpha}\{S^{\beta}(s^n)\}=S^{\alpha+\beta}(s_n)$, in particular $S^{-\alpha}\{S^{\alpha}(s^n)\}=S^0(s_n)=s_n$. Also $S^1(s_n)=s_0+s_1+\cdots+s_n$, and $A_n^{\alpha}=\binom{\alpha+n}{n}=(-1)^n\binom{-\alpha-1}{n}=S^{\alpha}(1)$.

Since we usually write

$$C^{\beta}(s_n) = \frac{S^{\beta}(s_n)}{A_n^{\beta}} \quad (\beta > -1)$$

for the n-th Cesàro mean of order β of a sequence s_ν (defined for $\nu = 0, 1, 2, \cdots$), we get

$$C^{\beta}(s^n) \frac{A_n^{\beta}}{n^{\beta}} = 0(1), \quad n \to \infty \quad (\beta > -1).$$

From Stirling's Theorem, as is well known, we have

$$A_n^{\beta} = \frac{n^{\beta}}{\Gamma(\beta+1)} + \sum_{\nu=1}^{p} C_\nu n^{\beta-\nu} + 0(n^{\beta-p-1}),$$

and it follows that

$$\lim_{n\to\infty} \frac{A_n^{\beta}}{n^{\beta}} = \frac{1}{\Gamma(\beta+1)}.$$

Therefore $C^{\beta}(s_n) = 0(1)$, $n \to \infty$ $(\beta > -1)$.
Hence, $s_n = 0(1)$ (C, β), $n \to \infty$ $(\beta > -1)$, that is,

$$\sum_{n=0}^{\infty} a_n = 0(C, \beta) \; (\beta > -1).$$

Applying the theorem proved by A. F. Anderson in [4][2], if

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} A_n^0 a_n = 0(C, \beta; \beta > -1),$$

then $\sum_{n=0}^{\infty} A_n^1 (a_n - a_{n+1}) = 0(C, \beta+1)$, that is, we have

$$\sum_{n=0}^{\infty} (n+1)(a_n - a_{n+1}) = 0\,(C, \beta+1).$$

Taking the n-th partial sum of the series $\sum_{n=0}^{\infty} (n+1)(a_n - a_{n+1})$, we obtain

$$\begin{aligned}\sum_{\nu=0}^{n} (\nu+1)(a_\nu - a_{\nu+1}) &= (a_0 - a_1) + 2(a_1 - a_2) + \cdots + (n+1)(a_n - a_{n+1}) \\ &= a_0 + a_1 + \cdots + a_n - (n+1)\,a_{n+1} \\ &= s_n - (n+1)\,a_{n+1}.\end{aligned}$$

Hence, we get

$$s_n - (n+1)\,a_{n+1} = 0(1)(C, \beta+1), \quad n \to \infty.$$

From $\sum a_n = 0\,(C, \beta)$ $(\beta > -1)$ and the inclusion theorem for Cesàro means, we have

$$\sum_{n=0}^{\infty} a_n = 0\,(C, \beta+1)(\beta > -1),$$

that is

$$s_n = 0(1)(C, \beta+1), \quad n \to \infty.$$

Therefore we have

$$(n+1)\,a_{n+1} = 0(1)(C, \beta+1), \quad n \to \infty,$$

2) Theorem of Anderson.
For ρ real $(\rho \neq 0)$, $k > -1$, if $\sum_{n=0}^{\infty} A_n^{\rho-1} a_n$ is summable (C, k), then $\sum_{n=0}^{\infty} A_n^{\rho} (a_n - a_{n+1})$ is summable $(C, k+1)$ with the same sum.

that is,

$$na_n = 0(1)(C, \beta+1), \quad n\to\infty.$$

As $\beta>-1$, we get

$$na_n = 0(1)(H, \beta+1), \quad n\to\infty.$$

Therefore by the well known Tauberian theorem, from

$$na_n = 0(1)\ (H, \beta+1), \quad n\to\infty$$

and our other hypothesis $s_n\to s(A)$ $(n\to\infty)$, we have

$$s_n\to s(H, \beta), \quad n\to\infty,$$

that is,

$$s_n\to s(H, -k), \quad n\to\infty.$$

Thus our theorem is proved.

Next, we will prove the case of a fractional number k for $k>1$.

Theorem 2. If $\{s_n\}$ is summable (A) to s, and if, for a fractional number k, $k>1$.

$$\binom{n}{m} \Delta_m s_{n-m} = 0(1)\ (H, -l),$$

then $\{s_n\}$ is summable $(H, -k)$ to s, where $k=l+m$, l is a fractional number for $0<l<1$ and m is a positive integer.

Proof. By our assumption

$$\binom{n}{m} \Delta^m s_{n-m} = 0(1)\ (H, -l),$$

we have

$$\binom{n}{m} \Delta^m s_{n-m} = 0(1)\ (H, -l+l),$$

that is,

$$\binom{n}{m} \Delta^m s_{n-m} = 0(1).$$

From the other assumption, $\{s_n\}$ is summable (A) to s; and so, by the Jakimovski theorem, $\{s_n\}$ is summable $(H, -m)$ to s, or $\{h_n^{(-l-m)}\}$ is summable (H, l) to s, and consequently summable (A) to s too. Thus, from the proposition proved by O. Szász in [5][3] $\{h_n^{(-l)}\}$ is also summable (A) to s. Also from our hypothesis,

$$\binom{n}{m}\Delta^m s_{n-m} = 0(1)\ (H, -l),$$

and so, by the Jakimovski theorem, $\{h_n^{(-l)}\}$ is summable $(H, -m)$ to s. Hence, $\{s_n\}$ is summable $(H, -l-m)$ to s, that is, $\{s_n\}$ is summable $(H, -k)$. Thus our theorem is proved.

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3) Theorem of O. Szász. If $\{s_n\}$ is summable (A) to s and $\{t_n\}$ is a regular Hausdorff transform of $\{s_n\}$; then $\{t_n\}$ is summable (A) to s too.

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