計量s=∫（A_ix''^I+B)^<1/p>dtをもつn次元空間における曲線の共形論

On the Conformal Theory of Curves in an n-Dimensional Space with the Arc Length $s=\int(A_i x''^i+B)^{1/p}\, dt$

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叶 長太郎: 計量 $s=\int(A_i x''^i+B)^{1/p}\, dt$ をもつ n 次元空間における曲線の共形論

Introduction. The fundamental theory of the n-dimensional Kawaguchi space with the metrics $s=\int F^{1/p}\, dt$, where $F=A_i x''^i+B$, was developed by Prof. A. Kawaguchi [1], [2]. The general curve theory of the space was studied by S. Ide [3], [4]. In the previous paper [5] the present author has introduced a conformal theory in such a space.

In the present paper we shall develope a conformal curve theory in the space.

§ I. **Conformal parameter.** In the previous paper we have seen the following relations between the fundamental quantities:

$$
(1.1)\qquad
\begin{aligned}
\tilde{A}_i\left(x, \frac{dx}{dt}\right) &= e^{p\sigma} A_i\left(x, \frac{dx}{dt}\right),\\
\tilde{A}_{i(j)} &= e^{p\sigma}\left(A_{i(j)}+p\sigma_{(j)}A_i\right),\\
\tilde{G}_{ij} &= e^{p\sigma}\left\{G_{ij}+p\left(2\sigma_{(j)}A_i-\sigma_{(i)}A_j\right)\right\},\\
\tilde{A}_{ij} &= e^{p\sigma}\left(A_{ij}+p\sigma_j A_i\right),
\end{aligned}
$$

where the parameter t is a conformal parameter defined along the curve.

Instead of the parameter t, if we take the invariant parameter s defined as the arc length of a curve, then (1. 1) becomes

$$
(1.2)\qquad
\begin{aligned}
\tilde{A}_i\left(x, \frac{dx}{d\bar{s}}\right) &= e^{2\sigma} A_i\left(x, \frac{dx}{ds}\right), \qquad \tilde{B}\left(x, \frac{dx}{d\bar{s}}\right) = B\left(x, \frac{dx}{ds}\right),\\
\tilde{A}_{i(j)} &= e^{3\sigma}\left(A_{i(j)}+p\sigma_{(j)}A_i\right),\\
\tilde{A}_{ij} &= e^{2\sigma}\left(A_{ij}+p\sigma_j A_i\right),\\
\tilde{G}_{ij} &= e^{3\sigma}\left(G_{ij}+p(2\sigma_{(j)}A_i-\sigma_{(i)}A_j\right),
\end{aligned}
$$

Then the conformal base connection

$$
x^{|2|i} = \frac{d^2x^i}{ds^2} + 2H^i\left(x, \frac{dx}{ds}\right),
$$

where $H^i=\Gamma^i-Kx'^i$ and $K=G^{ij}\nabla_j A_i\Big/\left(\frac{p^2-3}{p(2p-3)}-G^{ij}A_{i(j)}\right)$, aer not conformal. Hence we have from (1. 2) the following transformation law under the conformal transformation

$$
(1.3)\qquad \tilde{x}^{|2|i} = e^{-2\sigma}\left(x^{|2|i}-\frac{d\sigma}{ds}\cdot\frac{dx^i}{ds}\right).
$$

Then we find, for the conformal differential of $x^{|2|i}$ along the curve $x^i=x^i(s)$,

$$\frac{\delta^*\tilde{x}^{|2|i}}{d\bar{s}}=e^{-3\sigma}\left\{\frac{\delta^* x^{|2|i}}{ds}+\left(2\left(\frac{d\sigma}{ds}\right)^2-\frac{d^2\sigma}{ds^2}\right)\frac{dx^i}{ds}-3\frac{d\sigma}{ds}x^{|2|i}\right\}.$$

If these equations be multiplied by A_i and summed for i, we obtain

$$\text{(1.4)}\qquad \tilde{Q}(\bar{s})=e^{-\sigma}\left\{Q(s)-\frac{d\sigma}{ds}\right\},$$

where

$$Q(s)=\frac{1}{2}A_i\frac{\delta^* x^{|2|i}}{ds}=-\frac{1}{3}\{G_{ij}x^{|2|i}x^{|2|j}+(2p-3)K\}.$$

Thus if we put

$$\varphi(t)=\left[\{t,s\}-\frac{1}{2}\left(2\frac{dQ}{ds}-Q^2\right)\right]\Big/\left(\frac{dt}{ds}\right)^2,$$

where $\{t, s\}$ denotes the Schwarizian derivative of t with respect to s, then from (1. 4) it. follows that the quantity $\varphi(t)$ is invariant under the conformal transformation, and consequently the parameter t defined by the equation $\varphi(t)=0$ is of conformal character.

§ 2. **Conformal are length.** If we put $x^{*|2|i}=x^{|2|i}-Q(s)\frac{dx^i}{ds}$, then by means of (1. 3) and (1. 4) the transformation law of $x^{*|2|i}$ under the conformal transformation is $\tilde{x}^{*|2|i}=e^{-2\sigma}x^{*|2|i}$. Further if we put $V^i=\frac{\delta^* x^{*|2|i}}{ds}-2Q(s)\frac{dx^i}{ds}$, we have $\tilde{V}^i=e^{-3\sigma}V^i$, where

$$\text{(2.1)}\qquad V^i=\frac{\delta x^{|2|i}}{ds}+\left\{-K_{(j)}x^{[2]j}-\frac{1}{3}\frac{\delta\nu}{ds}+\frac{2}{9}R\nu\right\}\frac{dx^i}{ds}-Rx^{[2]i}$$

$$\frac{\delta x^{[2]i}}{ds}=\frac{dx^{[2]i}}{ds}+\Gamma^i_{(j)}x^{[2]j},\quad R=A_i x^{[3]i},$$

$$\nu=R+3K.$$

Then we have $A_iV^i=0$.

In parallel with the above-mentioned equations, covariant representation of them is that

$$\tilde{S}^{[1]}_i=e^\sigma S^{[1]}_i,\qquad \tilde{S}^{[2]}_i=S^{[2]}_i$$

$$S^{[1]}_i=\frac{\delta^* A_i}{ds}+\frac{2}{3}(R-3K)A_i=A^{[1]}_i+\left(\frac{2}{3}R-K\right)A_i,$$

$$\text{(2.2)}\qquad S^{[2]}_i=\frac{\delta^* A^{[1]}_i}{ds}+\frac{1}{3}(R-3K)S^{[1]}_i=A^{[1]}_i+(R-K)A^{[1]}_i+TA_i-K_{(i)},$$

$$T=\frac{d}{ds}\left(\frac{2}{3}R-K\right)+\frac{1}{3}R\left(\frac{2}{3}R-K\right),$$

$$S^{[1]}_i x'^i=-1,\qquad S^{[2]}_i x'^i=0.$$

Now we shall assume that $V^i\neq 0$.

Thus from (2. 1) and (2. 2) the quantity $J(s)$ defined by

$$J(s)=(S^{[2]}_iV^i)^{1/3}$$

$$\text{(2.3)}\qquad =(-k_2\bar{k}_2-k_1K+RK_{(i)}x^{[2]i}-K_{(i)}x^{[3]i})^{1/3},\qquad [3]$$

is transformed into $\tilde{J}(\bar{s})=e^{-\sigma}J(s)$.

Hence we can see that the parameter $\kappa(s)$ defined by

$$\text{(2.4)}\qquad \kappa(s)=\int J(s)ds$$

is a conformal parameter defined along the curve. We shall call it conformal are length of the curve.

§ 3. Conformal covariant derivative and conformal Frenet's Formulas.

Let us consider a curve $x^i=x^i(\kappa)$ and assume that κ is a conformal parameter. If v^i is a relative conformal contravariant vector defined along the curve, that is, $\tilde{v}^i=e^{-\sigma}v^i$, then from (1.4) we obtain

$$\frac{Dv^i}{d\kappa}=\frac{\delta^* v^i}{d\kappa}+J^{-1}Qv^i$$

that the transformation law of $Dv^i/d\kappa$ under conformal transformation is $D\tilde{v}^i/d\kappa=e^{-\sigma}Dv^i/d\kappa$. We shall call $Dv^i/d\kappa$ the conformal covariant derivative of the relative conformal conformal formal contravariant vector v^i along the curve.

For a relative conformal covariant vector we can define conformal covariant derivative in the same way, that is,

$$\frac{Dv^i}{d\kappa}=\frac{\delta^* v_i}{d\kappa}-J^{-1}Qv_i$$

Now if we put $\overset{1}{\eta}{}^i=\dfrac{dx^i}{ds}$ and $\underset{1}{\lambda}_i=JA_i$, then $\overset{1}{\eta}{}^i$ and $\underset{1}{\lambda}_i$ is transformed into $\overset{1}{\tilde{\eta}}{}^i=e^{-\sigma}\overset{1}{\eta}{}^i$, and $\underset{1}{\tilde{\lambda}}_i=e^{\sigma}\underset{1}{\lambda}_i$ respectively. Then we have $\underset{1}{\lambda}_i D\overset{1}{\eta}{}^i/d\kappa=1$, $\dfrac{D\underset{1}{\lambda}_i}{d\kappa}\overset{1}{\eta}{}^i=-1$ and $\underset{1}{\lambda}_i\overset{1}{\eta}{}^i=0$.

Thus with the aid of the following relations:

$$\begin{array}{llllll}
(3.1)\quad \underset{1}{\lambda}_i\overset{1}{\eta}{}^i=0, & \underset{1}{\lambda}_i\overset{2}{\eta}{}^i=1, & \underset{1}{\lambda}_i\overset{3}{\eta}{}^i=0, & \cdots & \cdots, & \underset{1}{\lambda}_i\overset{n}{\eta}{}^i=0, \\
\underset{2}{\lambda}_i\overset{1}{\eta}{}^i=-1, & \underset{2}{\lambda}_i\overset{2}{\eta}{}^i=k^*, & \underset{2}{\lambda}_i\overset{3}{\eta}{}^i=0, & \cdots & \cdots, & \underset{2}{\lambda}_i\overset{n}{\eta}{}^i=0, \\
\vdots & & & & & \\
\underset{r}{\lambda}_i\overset{1}{\eta}{}^i=0, & \underset{r}{\lambda}_i\overset{2}{\eta}{}^i=0, & \cdots & \underset{r}{\lambda}_i\overset{r}{\eta}{}^i=1, & \cdots, & \underset{r}{\lambda}_i\overset{n}{\eta}{}^i=0, \\
\vdots & & & & & \\
\underset{n}{\lambda}_i\overset{1}{\eta}{}^i=0, & \underset{n}{\lambda}_i\overset{2}{\eta}{}^i=0, & \cdots & \cdots & \cdots, & \underset{n}{\lambda}_i\overset{n}{\eta}{}^i=1, \quad [3].
\end{array}$$

We arrive at the conformal Frenet formulaes

$$\begin{array}{ll}
(3.2)\quad \dfrac{D\overset{1}{\eta}{}^i}{d\kappa}=\overset{2}{\eta}{}^i, & \dfrac{D\underset{1}{\lambda}_i}{d\kappa}=\underset{2}{\lambda}_i, \\
\dfrac{D\overset{2}{\eta}{}^i}{d\kappa}=-k_1^*\overset{1}{\eta}{}^i-k^*\overset{2}{\eta}{}^i-\bar{k}_2^*\overset{3}{\eta}{}^i, & \dfrac{D\underset{2}{\lambda}_i}{d\kappa}=-\bar{k}_1^*\underset{1}{\lambda}_i+k^*\underset{2}{\lambda}_i+k_2^*\underset{3}{\lambda}_i, \\
\dfrac{D\overset{3}{\eta}{}^i}{d\kappa}=k_2^*\overset{1}{\eta}{}^i+\bar{k}^*\overset{4}{\eta}{}^i, & \dfrac{D\underset{3}{\lambda}_i}{d\kappa}=\bar{k}_2^*\underset{1}{\lambda}_i+k_3^*\underset{4}{\lambda}_i, \\
\dfrac{D\overset{r}{\eta}{}^i}{d\kappa}=-k_{r-1}^*\overset{r-1}{\eta}{}^i+\bar{k}_r^*\overset{r+1}{\eta}{}^i, & \dfrac{D\underset{r}{\lambda}_i}{d\kappa}=-\bar{k}_{r-1}^*\underset{r-1}{\lambda}_i+k^*\underset{r+1}{\lambda}_i, \\
\dfrac{D\overset{}{\underset{n}{\eta}}{}^i}{d\kappa}=-k_{n-1}^*\overset{n-1}{\eta}{}^i, & \dfrac{D\underset{n}{\lambda}_i}{d\kappa}=-\bar{k}_{n-1}^*\underset{n-1}{\lambda}_i, \\
\qquad (4\leqq r\leqq n-1), &
\end{array}$$

where

$$\frac{dk^*}{d\kappa}=k_1^*-\bar{k}_1^*,\qquad k_2^*\bar{k}_2^*=k^*(\bar{k}_1^*+k_1^*)-k^{*3}-\underset{1}{\lambda}{}_i^{(2)}\overset{1}{\eta}{}^{i(2)},$$

$$\bar{k}^* k_r^* = \sum_{s=3}^{r} (-1)^{s-1} \underset{s}{\lambda_i^{(1)}} \overset{i}{\eta}{}^{i(1)}, \qquad (3 \leqq r \leqq n-1), \quad [3].$$

In these equations the n vectors $\overset{1}{\eta}{}^i, \overset{2}{\eta}{}^i, \cdots, \overset{n}{\eta}{}^i$ form the associate conformal ennuple at a point of the curve.

Thus, making use of the quantity:

$$J^*(\kappa) = J\frac{d^2 J}{d\kappa^2} - \frac{3}{2}\left(\frac{dJ}{d\kappa}\right)^2 - \frac{1}{2}\left\{Q(\kappa) - 2J\frac{dQ}{d\kappa}\right\} \Big/ J^2,$$

$$Q(\kappa) = \frac{1}{p-6}\left\{J^{p-1}(G_{ij}(\kappa)\overset{2}{\eta}{}^i\overset{2}{\eta}{}^j + (2p-3)JK\right\},$$

the existence theorem in the conformal theory of a curve can be treated by the same method as that A. Fialkow [6].

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References

[1] A. Kawaguchi: Die Geometrie Des Integral $\int(A_i x''^i+B)^{1/p}dt$, Proc. Acade., 12 (1936), 205–208.

[2] A. Kawaguchi: Geometry in the an n–dimensional space with the arc length $s=\int(A_i x''^i+B)^{1/p}\,dt$, Trans. of the Amer. Math. Soc., 44 (1938), 158–167.

[3] S. Ide: On the theory of curves in an n–dimensional space with the metrics $s=\int(A_i x''^i+B)^{1/p}\,dt$, Tensor, Old Series, 9 (1949), 25–29.

[4] S. Ide: On the theory of curves in an n–dimensional space with the metrics $s=\int(A_i x''^i+B)^{1/p}\,dt$, II, Tensor, New Series, 2 (1952), 89–98.

[5] C. Kano: Conformal geometry in an n–dimensional space with the arc length $s=\int(A_i x''^i+B)^{1/p}\,dt$, Tensor, New Series, 5 (1956), 187–196.

[6] A. Fialkow: The conformal theory of curves, Trans. of the Amer. Soc., 51 (1942), 435–501.