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On the Hölder Mean of Negative Order (II)

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三浦白治： 負数次のヘルダー総和法について

1. Introduction. Amnon Jakimovski and C. T. Rajagopal proved the following Tauberian theorem for the Hölder mean of negative order $-k$ in [1], [2].

Theorem A (Theorem of Jakimovski in [1]). Let k be a fixed positive integer. A necessary and sufficient condition for $\{s_n\}$ to be summable (H, $-k$) to s is that $\{s_n\}$ should be summable (A) to s and

$$\lim_{n\to\infty}\binom{n}{k}\cdot\Delta^k s_{n-k}=o.$$

Theorem B (Theorem of Rajagopal in [2]). (a) If (i) $\{s_n\}$ is summable (A) to s, and if (ii) for a positive integer k,

$$n^k\Delta^k s_{n-k}=o(1),\ n\to\infty,$$

then $\{s_n\}$ is summable (H, $-k$) to s.

(b) Conditions (i) and (ii) are also necessary $\{s_n\}$ to be summable (H, $-k$) to s.

In this paper, we will extend the above result to the case of some fixed positive fractional number k.

Now, we shall denote, in this paper, by $\{h_n^{(\alpha)}\}$ or the (H, α) transform, where α is an arbitrary fixed real number, the Hölder transform of order α of $\{s_n\}$. Also we shall use later the following Lemma 1 in [1].

Lemma 1 in [1]. Let α be a real number and k a nonnegative integer ; then, for any sequence $\{s_n\}$,

$$\binom{n}{k}\Delta^k h_{n-k}^{(\alpha)}=\sum_{p=0}^{k}a_p^{(k)}\cdot h_n^{(\alpha-k+p)},$$

for $n=0,\ 1,\ 2,\cdots\cdots\cdots$.

2. An extension of Theorem A or Theorem B. By the application of the difference of fractional order in [3], we will make an extension of Theorem A or Theorem B in the case of a positive fractional number k.

Theorem 1. Let k be a fixed fractional number k for $0<k<1$. A necessary and sufficient condition for $\{s_n\}$ to be summable (H, $-k$) to s is that $\{s_n\}$ should be summable (A) to s and

$$n^k S^{-k}(s_n)^{1)} = o(1), \quad n \to \infty,$$

where $s_n = a_0 + a_1 + \cdots\cdots + a_n$.

We have already proved the sufficency part in the previous paper (I). In this paper (II), we will add the proof of the necessity part.

Proof of the necessity part. Let us now put $k = -\beta$. Then our hypothesis

$$s_n \to s(H, -k) \ (0 < k < 1)$$

is that

$$s_n \to s(H, \beta) \ (\beta > -1).$$

From our above hypothesis, by the well-known Tauberian theorem, $\{s_n\}$ is summable (A) to s and

$$na_n = o(1) \ (H, \beta + 1),$$

By the application of the Aberian transformation to both of the above terms, we have

$$\Delta n \cdot S^1(a_n)^{1)} = o(1) \ (H, \beta), \ n \to \infty,$$

that is,

$$\sum_{n=0}^{\infty} a_n = 0(H, \beta) \ (\beta > -1).$$

i. e.,

$$s_n = o(1) \ (H, \beta) \ (\beta > -1).$$

As $\beta > -1$, we get

$$s_n = o(1) \ (C, \beta), \ n \to \infty,$$

that is,

$$C^\beta(s_n) = o(1) \ (\beta > -1), \ n \to \infty.$$

Now, as already stated in the previous paper (I), from Stirling's theorem, we have

$$\lim_{n \to \infty} \frac{A_n^\beta}{n^\beta} = \frac{1}{\Gamma(\beta+1)} (\beta > -1),$$

and

$$C^\beta(s_n) = \frac{S^\beta(s_n)}{A_\beta^n} (\beta > -1).$$

1) The difference of fractional order in [3] is defined by the operation of "inverse summation" :

$$S^{-k}(s_n) = \sum_{\nu=0}^{n} A_{n-\nu}^{-k-1} \ S_\nu,$$

which reduces, in the case

$k = 1$, *to* $S^{-1}(s_n) = s_n - s_{n-1}$,

provided we write

$s_{-1}, = o$. Also

$S^1(s_n) = s_0 + s_1 + \cdots\cdots + s_n$, and

$A_n^\alpha = \binom{\alpha+n}{n} = (-1)^n \binom{-\alpha-1}{n} = S^\alpha(1)$.

Accordingly we have

$$\frac{S^{\beta}(s_n)}{A_n^{\beta}} \cdot \frac{A_n^{\beta}}{n^{\beta}} = o(1),\ n\to\infty(\beta>-1),$$

that is

$$n^{-\beta}S^{\beta}(s_n) = o(1),\ n\to\infty(\beta>-1),$$

i. e.,

$$n^{k}S^{-k}(s_n) = o(1),\ n\to\infty(0<k<1).$$

Thus our theorem is proved.

Next, we will prove the case of a fractional number k for $k>1$.

Theorem 2. Let k be a fixed fractional number for $k>1$. A necessary and sufficient condition for $\{s_n\}$ to be summable (H, $-$k) to s is that $\{s_n\}$ should be summable (A) to s and

$$\binom{n}{m}\Delta^{m}s_{n=m} = o(1)\ (H,\ -l),\ n\to\infty,$$

where $k=l+m$, l is a fractional number for $0<l<1$ and m is a positive integer.

Proof of the necessity part. From our assumption

$$s_n\to s(H,\ -k)\ (k>1),$$

the necessity of the Abel summability of $\{s_n\}$ to s is obvious.
Also the necessity of

$$\binom{n}{m}\Delta^{m}s_{n-m} = o(1)\ (H,\ -l),\ n\to\infty$$

will be proved as follows. From Lemma 1 in [1], we obtain

$$\binom{n}{m}\Delta^{m}h_{n-m}^{(-l)} = \sum_{p=o}^{m}a_p^{(m)}h_n^{(-l-m+p)} = \sum_{p=o}^{m}a_p^{(m)}h_n^{(-k+p)},$$

for n=0, 1, 2, ……… . From our assumption

$$\lim_{n\to\infty} h_n^{(-k)} = s,$$

we have

$$\lim_{n\to\infty} h_n^{(-k+p)} = s,$$

for $p=0,\ 1,\ 2,\ \cdots\cdots\cdots\cdots,m$. Also

$$\sum_{p=o}^{m}a_p^{(m)} = 0,\ a_o^{(m)} \neq 0,$$

for $m=0, 1, 2, \cdots\cdots\cdots\cdots$.[2] Therefore

$$\lim_{n\to 8} \binom{n}{m} \Delta^m h_{n-m}^{(-1)} = s\sum_{p=o}^{m} a_p^{(m)} = s \cdot o = o,$$

i. e.,

$$\binom{n}{m} \Delta^m s_{n-m} = o(1) \quad (H, \ -1), \ n \to \infty.$$

Thus our theorem is proved.

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References

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2. C. T. Rajagopal : Simplified proofs of "some Tauberian theorems" oj Jakimovski, Pacific Journal, 7 (1957) 955—960.
3. L. S. Bosauquet : An extension of a theorem of Andersen, Journal London Math. Soc., 25(1950) 72—80.
4. G. H. Hardy : Divergent series (Oxford, 1949).

2) The following results are shown in [1].

$$\binom{n}{k} \Delta^k s_{n-k} = \sum_{p=o}^{k} a_p^{(k)} h_n^{(-k+p)},$$

for $n=0, 1, 2, \cdots\cdots\cdots\cdots$;

$$\sum_{p=o}^{k} a_p^{(k)} = o \ ; \quad a_o^{(k)} \neq o$$

for $k=0, 1, 2, \cdots\cdots\cdots\cdots$