特殊河口空間におけるextremal曲線に就いて

ON EXTREMAL CURVE IN A SPECIAL KAWAGUCHI SPACE II.

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川口俊一：特殊河口空間における extremal 曲線に就いて

§ 0. Introduction.

The theory of n-dimensional metric space that the arc length of a curve $x^i = x^i(t)$ is given by the integral $s = \int \left\{ A_i(x, x') x''^i + B \right\}^{\frac{1}{p}} dt$ was deveroped by Prof. A. Kawaguchi [1], [2], and afterward studied by many others. In previous paper [4] the present auther dealed with a relation between extremal curve and autoparallel curve in such a space. Recentry, M. Okumura [5] has introduced a conformal curvature tensor and obtained interesting results for two curves.

In present paper we shall introduce Weyl's tensor in the space which studied already in a Finsler space and study a relation between Weyl's tensor and conformal curvature tensor. We shall give the Weyl's tensor in §1 and the conformal curvature tensor in §2. In §3 we shall consider the relation between Weyl's tensor and conformal curvature tensor.

The notions and technics employed here without explanations are the same as those of Prof. A. Kawaguchi.

§ 1. Weyl's tensor.

For a contravariant vector v^i we can define the covariant differential by

$$\delta v^i \equiv dv^i + \Gamma^i_{(j)(k)} v^j dx^k = dx^j \nabla_j v^i + \delta x'^j \nabla'_j v^i,$$

where

$$\delta x'^i = dx'^i + \Gamma^i_{(j)} dx^j, \quad \nabla_j v^i = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^k} \Gamma^k_{(j)} + \Gamma^i_{(k)(j)} v^k, \quad \nabla'_j v^i = \frac{\partial v^i}{\partial x'^j}.$$

Then we find the curvature tensor as follows :

$$(\nabla_j \nabla_k - \nabla_k \nabla_j) v^i = - R_{jkl}^{\cdot\cdot\cdot i} v^l + K_{jk}^{\cdot\cdot l} \nabla'_l v^i,$$

$$(\nabla_j \nabla'_k - \nabla'_k \nabla_j) v^i = - B_{jkl}^{\cdot\cdot\cdot i} v^l,$$

where

$$B_{jkl}^{\cdot\cdot\cdot i} = \Gamma^i_{(j)(k)(l)},$$

$$(1.1) \quad \begin{aligned} R_{jj}^{\cdot\cdot i} = \frac{\partial \Gamma^i_{(l)(j)}}{\partial x^k} - \frac{\partial \Gamma^i_{(l)(k)}}{\partial x^j} + \Gamma^h_{(l)(j)} \Gamma^i_{(k)(h)} - \Gamma^h_{(l)(k)} \Gamma^i_{(j)(h)} \\ + \Gamma^h_{(j)} \Gamma^i_{(l)(k)(h)} - \Gamma^h_{(k)} \Gamma^i_{(l)(j)(h)}, \end{aligned}$$

$$K_{jk}^{\cdot\cdot i} = \frac{\partial \Gamma^i_{(j)}}{\partial x^k} - \frac{\partial \Gamma^i_{(k)}}{\partial x^j} + \Gamma^h_{(j)} \Gamma^i_{(k)(h)} - \Gamma^h_{(k)} \Gamma^i_{(j)(h)}.$$

Then, the following relations are satisfied by those curvature tensors.

$$R_{j\dot{k}\dot{i}}{}^{l} + R_{\dot{k}j\dot{i}}{}^{l} = 0, \qquad R_{j\dot{k}\dot{i}}{}^{l} + R_{\dot{k}\dot{i}j}{}^{l} + R_{\dot{i}j\dot{k}}{}^{l} = 0,$$

$$K_{j\dot{k}}{}^{l} = R_{j}{}^{\cdot}{}_{\dot{k}}{}^{\cdot}{}_{\dot{i}}{}^{l} x'^{i}, \qquad R_{j\dot{k}\dot{i}}{}^{l} = K_{j\dot{k}}{}^{l}{}_{(i)}\,. \tag{1.2}$$

We define the new function $\bar{\Gamma}^{l}(x, x')$ by the equations

$$\bar{\Gamma}^{l}(x, x') = \Gamma^{l} - P(x, x')\, x'^{l}, \tag{1.3}$$

where $P(x, x')$ is an arbitrary scalar function positive homogeneous of the first degree in the x'^{i}. Then, differentiating (1.3) in x'^{j} and x'^{k}, we have

$$\bar{\Gamma}^{l}_{(j)(k)} = \Gamma^{l}_{(j)(k)} - P_{(k)}\,\delta^{l}_{j} - P_{(j)}\,\delta^{l}_{k} - P_{(j)(k)}\,x'^{l}. \tag{1.4}$$

Putting $H^{l}_{k} \equiv K_{jk}^{\cdot\cdot l}\, x'^{j}$, we can define curvature tensor $\bar{H}^{l}_{k} \equiv \bar{K}_{jk}^{\cdot\cdot l}\, x'^{j}$ with respect to the new functions $\bar{\Gamma}^{l}$:

$$\bar{H}^{l}_{k} \equiv \bar{K}_{jk}^{\cdot\cdot l}\, x'^{j} = 2\,\frac{\partial \bar{\Gamma}^{l}}{\partial x^{k}} - \frac{\partial \bar{\Gamma}^{l}_{(k)}}{\partial x^{j}}\, x'^{j} + 2\,\bar{\Gamma}^{l}_{(k)(h)}\bar{\Gamma}^{h} - \bar{\Gamma}^{l}_{(h)}\,\bar{\Gamma}^{h}_{(k)}\,. \tag{1.5}$$

From (1.3) and (1.5), we have

$$\bar{H}^{l}_{k} = H^{l}_{k} - x'^{l}\left(2\,\frac{\partial P}{\partial x^{k}} - \frac{\partial P_{(k)}}{\partial x^{h}}\, x'^{h} + 2\,P_{(k)(h)}\,\Gamma^{h} - P_{(h)}\Gamma^{h}_{(k)} + P\,P_{(k)}\right) + \delta^{l}_{k}\left(\frac{\partial P}{\partial x^{h}}\, x'^{h} - 2\,P_{(h)}\,\Gamma^{h} + P^{2}\right). \tag{1.6}$$

Making use of H, $\bar{H}$, which defined by $\bar{H} \equiv \frac{1}{n-1}\,\bar{H}^{i}_{i}$, $H \equiv \frac{1}{n-1} H^{i}_{i}$, from (1.6) we have

$$\bar{H} = H - \left(2\,P_{(h)}\,\Gamma^{h} - \frac{\partial P}{\partial x^{h}}\, x'^{h} - P^{2}\right). \tag{1.7}$$

We differentiate the equation with respect to x'^{r} that combined (1.6) and (1.7), and contract i and r. Moreover, combining this obtained equation, (1.6) and (1.7), we have

$$\frac{1}{n+1}\left(\frac{\partial \bar{H}^{a}_{k}}{\partial x'^{a}} - \frac{\partial \bar{H}}{\partial x'^{k}}\right) x'^{l} - \left(\bar{H}^{l}_{k} - \bar{H}\,\delta^{l}_{k}\right) = \frac{1}{n+1}\left(\frac{\partial H^{a}_{k}}{\partial x'^{a}} - \frac{\partial H}{\partial x'^{k}}\right) x'^{l} - \left(H^{l}_{k} - H\delta^{l}_{k}\right).$$

Hence, the tensor defined by

$$W^{l}_{k} \equiv H^{l}_{k} - H\,\delta^{l}_{k} - \frac{1}{n+1}\left(\frac{\partial H^{a}_{k}}{\partial x'^{a}} - \frac{\partial H}{\partial x'^{k}}\right) x'^{l} \tag{1.8}$$

is invariant under the projective transformation, and call it Weyl's tensor. Thus defined Weyl's tensor satisfies the following relations

$$W^{l}_{l} = 0\,, \qquad W^{l}_{k}\, x'^{k} = 0\,. \tag{1.9}$$

Differetiating (1.9) with respect to x'^{r}, and then contract i and r, we have

$$\frac{\partial W^{l}_{k}}{\partial x'^{l}} = 0\,. \tag{1.10}$$

§ 2. Conformal curvature tensor.

In [5], M. Okumura defined his conformal curvature tensor of the first kind as follows :

$$C^{*}{}_{jkl}^{\cdots i} = R^{*}{}_{jkl}^{\cdots i} - \frac{\delta^{i}_{l}}{n+1}\, S^{*}_{jk} + \frac{\delta^{i}_{k}}{n-1}\left(R^{*}_{jl} - \frac{1}{n+1}\, S^{*}_{lj}\right) - \frac{\delta^{i}_{j}}{n-1}\left(R^{*}_{kl} - \frac{1}{n+1}\, S^{*}_{lk}\right), \tag{2.1}$$

where

(2.2) $$R^{*\cdot\cdot\cdot i}_{jkl} = \frac{\partial \Pi^i_{lj}}{\partial x^k} - \frac{\partial \Pi^i_{lk}}{\partial x^j} + \Pi^h_{lj}\,\Pi^i_{kh} - \Pi^h_{lk}\,\Pi^i_{jh} + \Pi^h_j\,\Pi^i_{lk(h)} - \Pi^h_k\,\Pi^i_{lj(h)}\,,$$

(2.3) $$\Pi^i_{jk} = \Gamma^i_{(j)(k)} - \frac{1}{n+1}\,\Gamma^h_{(j)(k)(h)}\,x'^i\,,\quad \Pi^h_j = \Pi^h_{jk}\,\Pi x'^k$$

(2.4) $$R^*_{kl} = R^{*\cdot\cdot\cdot a}_{akl}\,,\quad R^*_{jk} = R^{*\cdot\cdot\cdot a}_{jka}$$

Thus defined conformal curvature tensor is homogeneous function of degree zero with respect to x'^i and satisfies the following relations :

$$C^{*\cdot\cdot\cdot a}_{akl} = C^{*\cdot\cdot\cdot a}_{kal} = C^{*\cdot\cdot\cdot a}_{kla} = 0\,,$$

(2.5) $$C^{*\cdot\cdot\cdot a}_{jkl\,(a)} = 0\,,$$

$$C^{*\cdot\cdot\cdot i}_{jkl} + C^{*\cdot\cdot\cdot i}_{kjl} = 0$$

§ 3. Relation between W^i_k and $C^{*\cdot\cdot\cdot i}_{jkl}$.

From (1.8), $H^i_k = K^{\cdot\cdot i}_{jk}\,x'^j$ and (1.2), we have

$$W^i_k = R^{\cdot\cdot\cdot i}_{jkl}\,x'^j\,x'^l - \frac{\delta^i_k}{n-1}\,R^{*\cdot\cdot\cdot a}_{jal}\,x'^j\,x'^l - \frac{1}{n+1}\,\frac{\partial}{\partial x'^a}\,(R^{\cdot\cdot\cdot a}_{jkl})\,x'^j\,x'^l\,x'^i$$
$$- \frac{1}{n+1}\,R^{\cdot\cdot\cdot a}_{akl}\,x'^l\,x'^i - \frac{1}{n+1}\,R^{\cdot\cdot\cdot a}_{lak}\,x'^l\,x'^i + \frac{1}{n^2-1}\,\frac{\partial}{\partial x'^k}\,(R^{\cdot\cdot\cdot a}_{jal})\,x'^j\,x'^l\,x'^i$$
$$+ \frac{1}{n-1}\,R^{\cdot\cdot\cdot a}_{kal}\,x'^l\,x'^i + \frac{1}{n^2-1}\,R^{\cdot\cdot\cdot a}_{lak}\,x'^l\,x'^i.$$

From (1.1) and that $\Gamma^i_{(j)(k)}$ are homogeneous of degree zero with respect to x'^i, we have $\frac{\partial}{\partial x'^a}\,(R^{\cdot\cdot\cdot a}_{jkl})\,x'^j\,x'^l = 0$, $\frac{\partial}{\partial x'^k}\,(R^{\cdot\cdot\cdot a}_{jal})\,x'^j\,x'^l = 0$ ⊙

Therefore we have

(3.1) $$W^i_k = R^{\cdot\cdot\cdot i}_{jkl}\,x'^j\,x'^l - \frac{\delta^i_k}{n-1}\,R^{\cdot\cdot\cdot a}_{jal}\,x'^j\,x'^l\,\frac{1}{n+1}\,R^{\cdot\cdot\cdot a}_{akl}\,x'^l\,x'^i -$$
$$- \frac{1}{n+1}\,R^{\cdot\cdot\cdot a}_{jka}\,x'^l\,x'^i + \frac{1}{n^2-1}\,R^{\cdot\cdot\cdot a}_{kal}\,x'^l\,x'^i + \frac{1}{n^2-1}\,R^{\cdot\cdot\cdot a}_{lak}\,x'^l\,x'^i.$$

On the other hand, making use of (2.2) and (2.3), we have after some calculations

(3.2) $$R^{*\cdot\cdot\cdot i}_{jkl} = R^{\cdot\cdot\cdot i}_{jkl} - \frac{x'^i}{n+1}\Big[\frac{\partial \Gamma^h_{(l)(j)(h)}}{\partial x^k} - \frac{\partial \Gamma^h_{(l)(k)(h)}}{\partial x^j} + \Gamma^a_{(l)(j)}\,\Gamma^h_{(k)(a)(h)}$$
$$+ \Gamma^a_{(l)(k)}\,\Gamma^h_{(j)(a)(h)} + \Gamma^a_{(j)}\,\Gamma^h_{(l)(k)(h)(a)} - \Gamma^a_{(k)}\,\Gamma^h_{(l)(j)(h)(a)}\Big].$$

Contracting i and j in (3.2), and making use of $\Gamma^i_{(j)(k)(i)}\,x'^i = 0$, we have

(3.3) $$R^*_{kl} = R^{*\cdot\cdot\cdot a}_{akl}$$
$$= R^{\cdot\cdot\cdot a}_{akl} - \frac{1}{n+1}\Big[-x'^a\,\frac{\partial \Gamma^h_{(l)(k)(h)}}{\partial x^a} + \Gamma^b_{(l)}\,\Gamma^h_{(k)(b)(h)}$$
$$+ 2\,\Gamma^b\,\Gamma^h_{(l)(k)(h)(b)} + \Gamma^b_{(k)}\,\Gamma^h_{(l)(b)(h)}\Big]\,,$$

and moreover if we contract $x'^k\,x'^l$, we have

(3.4) $$R = R^*_{kl}\,x'^k\,x'^l = R^{\cdot\cdot\cdot a}_{akl}\,x'^k\,x'^l$$

Contracting i and l, we have because of $\Gamma^h_{(b)(k)(h)} + x'^a\,\Gamma^h_{(a)(k)(h)(b)} = 0$,

$$S^*_{jk} = R^{*\cdot\cdot\cdot a}_{jka} = R^{\cdot\cdot\cdot a}_{jka}$$ ⊙

Substituting (3.2), (3.3) and (3.5) in (2.1), it follows

$$C^{*\cdots l}_{jkl} = R^{\cdots l}_{jkl} - \frac{\delta^l_l}{n+1} R^{\cdots a}_{jka} + \frac{\delta^l_k}{n-1} R^{\cdots a}_{ajl} - \frac{\delta^l_k}{n^2-1} R^{\cdots a}_{lja} - \frac{\delta^l_j}{n-1} R^{\cdots a}_{akl}$$

$$+ \frac{\delta^l_j}{n^2-1} R^{\cdots a}_{lka} - \frac{x'^l}{n+1} \Big[\frac{\partial \Gamma^h_{(l)(j)(h)}}{\partial x^k} - \frac{\partial \Gamma^h_{(l)(k)(h)}}{\partial x^j} + \Gamma^a_{(l)(j)} \Gamma^h_{(k)(a)(h)}$$

$$- \Gamma^a_{(l)(k)} \Gamma^h_{(j)(a)(h)} + \Gamma^a_{(j)} \Gamma^h_{(l)(k)(h)(a)} - \Gamma^a_{(k)} \Gamma^h_{(l)(j)(h)(a)} \Big]$$

$$- \frac{\delta^l_k}{n^2-1} \Big[- x'^a \frac{\partial \Gamma^h_{(l)(j)(h)}}{\partial x^k} + \Gamma^a_{(l)} \Gamma^h_{(j)(a)(h)} + 2\,\Gamma^a \Gamma^h_{(l)(j)(h)(a)} + \Gamma^a_{(j)} \Gamma^h_{(l)(a)(h)} \Big]$$

$$- \frac{\delta^l_j}{n^2-1} \Big[- x'^a \frac{\partial \Gamma^h_{(l)(k)(h)}}{\partial x^a} + \Gamma^a_{(l)} \Gamma^h_{(k)(a)(h)} + 2\,\Gamma^a \Gamma^h_{(l)(k)(h)(a)} + \Gamma^a_{(k)} \Gamma^h_{(l)(a)(h)} \Big].$$

Multiplying x'^p x'^q to (3.6) and contracting p, q and j, k respectively, and making use of (1.2), we have

$$(3.7) \qquad C^{*\cdots l}_{jkl} x'^l x'^l = R^{\cdots l}_{jkl} x'^j x'^l - \frac{1}{n+1} R^{\cdots a}_{jka} x'^j x'^l + \frac{\delta^l_k}{n-1} R^{\cdots a}_{ajl} x'^j x'^l$$

$$+ \frac{1}{n^2-1} R^{\cdots a}_{lka} x'^l x'^l - \frac{1}{n-1} R^{\cdots a}_{akl} x'^l x'^l .$$

Taking (3.7) from (3.1), and making use of (1.2), we have

$$W^l_k - C^{*\cdots}_{jkl} x'^j x'^l = \frac{2}{n^2-1} R^{\cdots p}_{akl} x'^l x'^l + \frac{1}{n^2-1} \Big[R^{\cdots a}_{kal} + R^{\cdots a}_{lak} + R^{\cdots a}_{lka} \Big] x'^l x'^l$$

$$= \frac{2}{n^2-1} \Big[R^{\cdots a}_{akl} + R^{\cdots a}_{kal} \Big] x'^l x'^l$$

$$= 0 .$$

Accordingly we have

Theorem. *Let* W^l_k *be a tensor defined by* (3.1) *and let* $C^{*\cdots l}_{jkl}$ *be a tensor defined by* (2.1). *Then there exist the following relation among them :*

$$W^l_k = C^{*\cdots l}_{jkl} \; x'^j x'^l ,$$

In previous paper [4], we pointed out that the condition in order that an affine path be extremal curve is $A_l K_{j\,k}^{\cdot\,\cdot\,l} x'^j = O$. Multiplying A_l to (1.8) and contracting l and i , we have

$$A_l K_{j\,k}^{\cdot\,\cdot\,l} x'^j = A_l W^l_k + A_k H.$$

Thus we have

Theorem. *In the special Kawaguchi space with* $W^l_k = 0$, *a condition that an affine path be always extremal curve, is that* H *be equal to zero.*

Accordingly our condition coincides with that of M. Okumura.

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