A Property of N-function

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佐 々 木 幸 一 : N－函数の性質

M. A. Krasnosel'skii and Ya. B. Rutickii have considered in their investigation [1] the classes of N-functions in connection with the theory of Orlicz spaces. In this book several criteria are defined, namely Δ_2-, Δ'-, Δ_3-, and the Δ^2-condition. The purpose of this note is to difine a criterion for N-functions by a limiting process, and discuss the relations of this criterion to some other conditions.

Let function $p(t)$ $(0 \leqq t < \infty)$ be nondecreasing, continuous from the right, and satisfy the condition $p(+0)=0$, $p(t)>0$ for $t>0$, and $p(\infty)=\infty$.

Then the convex function

$$M(u) = \int_0^{|u|} p(t)dt \tag{1}$$

is called N-function. For the function $q(s) = \sup_{p(t) \geqq s} t$ $(0 \leqq s < \infty)$,

the function

$$N(v) = \int_0^{|v|} q(s)\,ds \tag{2}$$

is also a N-function, which is called the N-function complementary to $M(u)$.

For two N-functions $M_1(u)$ and $M_2(u)$, we write $M_1(u) \succ M_2(u)$ if

$$M_1(ku) \geqq M_2(u) \qquad (u \geqq u_0) \tag{3}$$

for some positive constants k and u_0. We write $M_1(u) \sim M_2(u)$ and say they are equivalent, if $M_1(u) \succ M_2(u)$ and $M_2(u) \succ M_1(u)$ hold.

The N-function (1) is even, positive for $u \neq 0$, and satisfies

$$\frac{1}{2} up\left(\frac{1}{2}u\right) \leqq M(u) \leqq up(u) \qquad (u>0). \tag{4}$$

It was shown in [1] that:

Lemma 1. *If there exist constants k, $u_0 > 0$ such that*

$$p_1(u) \leqq p_2(ku) \qquad (u \geqq u_0),$$

then the N-functions

$$M_1(u) = \int_0^{|u|} p_1(t)\,dt \quad \text{and} \quad M_2(u) = \int_0^{|u|} p_2(t)\,dt$$

satisfy the relation $M_1(u) \prec M_2(u)$.

The N-function (1) is said to satisfy the Δ_3-condition, if there exist two positive constants k and u_0, for which

$$M(ku) \geqq uM(u) \qquad (u \geqq u_0) \tag{5}$$

It is well-known, if $M(u)$ satisfies Δ_3-condition, then $M(u) \succ u^n$ for natural numbers $n \geqq 2$.

We consider the condition for the N-function $M(u)$

$$\lim_{u\to\infty}\frac{M(ku)}{M(u)}=\infty \qquad \text{for some } k>1. \tag{6}$$

If $M(u)$ satisfies this, and $M(u)\sim M_1(u)$ for N-function $M_1(u)$, then $M_1(u)$ also satisfies (6). In fact, under these conditions there exist constants $k>1$, $0<a<1$ and $b>1$ such that

$$\lim_{u\to\infty}\frac{M(ku)}{M(u)}=\infty \quad \text{and} \quad M_1(au)\leqq M(u)\leqq M_1(bu)$$

for sufficiently large u, therefore

$$\frac{M_1(kbu)}{M_1(au)}\geqq\frac{M(ku)}{M(u)}$$

hold for large u, and this shows that $M_1(u)$ satisfies condition (6).

Theorem 1. *A necessary and sufficient condition that the N-function $M(u)$ satisfy the condition* (6) *is that the N-function $N(v)$ complementary to $M(u)$ satisfy*

$$\inf_{\varepsilon>0}\ \varliminf_{v\to\infty}\frac{N(\varepsilon v)}{\varepsilon N(v)}>0. \tag{7}$$

Proof. It $M(u)$ satisfies condition (6), then for any $\varepsilon>0$,

$$M(ku)\geqq\frac{k}{\varepsilon}M(u) \tag{8}$$

for large u. Since N-functions which are complementary to $M(ku)$ and $\frac{k}{\varepsilon}M(u)$ respectively, are given in the forms of $N\left(\frac{v}{k}\right)$ and $\frac{k}{\varepsilon}N\left(\frac{\varepsilon v}{k}\right)$, it follows from (8) that

$$\frac{N\left(\frac{\varepsilon v}{k}\right)}{\varepsilon N\left(\frac{v}{k}\right)}\geqq\frac{1}{k}$$

for large v. Therefore

$$\varliminf_{v\to\infty}\frac{N(\varepsilon v)}{\varepsilon N(v)}\geqq\frac{1}{k}$$

for any $\varepsilon>0$, and

$$\inf_{\varepsilon>0}\ \varliminf_{v\to\infty}\frac{N(\varepsilon v)}{\varepsilon N(v)}\geqq\frac{1}{k}.$$

The necessity of the condition is proved. The sufficiency is proved analogously, thus proving the the theorem.

If a N-function $M(u)$ satisfies the Δ_3-condition, then there exist $k>1$ and $u_0>0$ such that

$$\frac{M(ku)}{M(u)}\geqq u \qquad (u\geqq u_0)$$

and condition (6) is satisfied. The converse is not true. In fact, the N-function $M(u)$ for which p. p. $M(u)=u^{\sqrt{\log u}}$ satisfies the condition (6), but not Δ_3-condition.

lemma 2. *In order that N-function $M(u)=\int_0^{|u|}p(t)\,dt$ satisfy the condition* (6), *it is necessary and sufficient that $p(t)$ satisfy* (6), *namely for some $k>1$*

$$\lim_{u\to\infty}\frac{p(ku)}{p(u)}=\infty \tag{9}$$

holds.

Proof. If $\lim_{u\to\infty}\frac{M(ku)}{M(u)}=\infty$ for some $k>1$, then

$$\frac{M(ku)}{M(u)}\leqq\frac{kup(ku)}{\frac{1}{2}up\left(\frac{1}{2}u\right)}=2k\frac{p(ku)}{p\left(\frac{1}{2}u\right)}$$

by (4), therefore

$$\lim_{u\to\infty}\frac{p(2ku)}{p(u)}=\infty.$$

Sufficiency is proved analogously.

Theorem 2. *If the N-function $M(u)$ satisfies the condition (6), then*

$$M(u)>u^n$$

for $n\geqq 2$.

Proof. Suppose that $\lim_{u\to\infty}\frac{M(ku)}{M(u)}=\infty$. For any natural number $n\geqq 2$ there exists $u_0>0$ such that

$$\frac{M(ku)}{M(u)}\geqq k^{2n}\qquad(u\geqq u_0).$$

For any $s\geqq k$, we can take natural number i such that $k^{i+1}>s\geqq k^i$. Then

$$k^2\geqq k^{\frac{i+1}{i}}>s^{\frac{1}{i}}\geqq k,$$

and for any $u\geqq u_0$,

$$\frac{M(s^{\frac{1}{i}}u)}{M(u)}\geqq\frac{M(ku)}{M(u)}\geqq k^{2n}=(k^2)^n>(s^{\frac{1}{i}})^n=\frac{(s^{\frac{1}{i}}u)^n}{u^n}$$

i. e.

$$\frac{M(s^{\frac{1}{i}}u)}{(s^{\frac{1}{i}}u)^n}\geqq\frac{M(u)}{u^n}\qquad(u\geqq u_0).$$

Then we have

$$\frac{M(su_0)}{(su_0)^n}=\frac{M((s^{\frac{1}{i}})^iu_0)}{((s^{\frac{1}{i}})^iu_0)^n}\geqq\frac{M((s^{\frac{1}{i}})^{i-1}u_0)}{((s^{\frac{1}{i}})^{i-1}u_0)^n}\geqq\cdots\cdots\geqq\frac{M(s^{\frac{1}{i}}u_0)}{(s^{\frac{1}{i}}u_0)^n}\geqq\frac{M(u_0)}{u_0{}^n}.$$

From this we obtain

$$M(su_0)\geqq M(u_0)s^n\qquad(s\geqq k),$$

i. e.

$$M(u)>u^n.$$

The theorem is proved.

The converse of this theorem is not true. If we put

$$p(u)=\begin{cases}u & \text{if } u\in[0,1)\\ (i\,!)^i & \text{if } u\in[(i-1)\,!,\ i\,!)\quad(i=2,3,\cdots\cdots),\end{cases}$$

we have N-function $M(u)=\int_0^{|u|}p(t)dt$. For any $k>1$, if we take natural number $n_0>k$, then

$$n\,!<kn\,!<n_0n\,!<(n+1)\,!$$

for $n \geqq n_0$, and

$$\lim_{u\to\infty} \frac{p(ku)}{p(u)} = \infty$$

cannot be satisfied because

$$\frac{p(kn!)}{p(n!)} = 1 \qquad (n \geqq n_0).$$

Using lemma 2, it can be seen that the condition (6) is not satisfied by $M(u)$.

On the other hand, for any $u \geqq n!$ where n is fixed, if we take m such that $(m+1)! > u \geqq m!$, then

$$p(u) = \{(m+1)!\}^{m+1} > \{(m+1)!\}^n > u^n,$$

therefore $M(u) > u^{n+1}$ by lemma 1.

Reference

[1] M. A. Krasnosel'skii and Ya. B. Rutickii, Convex functions and Orlicz spaces, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958.